Solutions

This test consists of three exercises. *Explain your answers.* The available points for each item are written in the margin.

[8] **1.** Consider the OCaml functions

```
let square x = x * x

let succ x   = x + 1
```

Give every computation step of the function call `square(succ 3)`, using the rightmost outermost reduction strategy.

Solution.

```
square(succ 3) → (succ 3) * (succ 3)
               → (succ 3) * (3 + 1)
               → (succ 3) * 4
               → (3 + 1) * 4
               → 4 * 4
               → 16
```

[8] **2.** Using the OCaml function

```
let rec map f = function []    -> []
                       | x::xs -> f x :: map f xs
```

prove that mapping two functions f and g one after another yields the same result as mapping the combined function $g \circ f$ (where $(g \circ f)\ x \stackrel{\text{def}}{=} g(f\ x)$), i.e., show

$$\texttt{map}\ g\ (\texttt{map}\ f\ xs) = \texttt{map}\ (g \circ f)\ xs$$

by induction over xs.

Solution. We show the property $P(xs) = \big(\texttt{map}\ g\ (\texttt{map}\ f\ xs) = \texttt{map}\ (g \circ f)\ xs\big)$.

Base Case ($xs = \texttt{[]}$). $P(\texttt{[]})$ is shown by the derivation:

$$\begin{aligned} \texttt{map}\ g\ (\texttt{map}\ f\ \texttt{[]}) &= \texttt{map}\ g\ \texttt{[]} && (\text{def. of } \texttt{map}) \\ &= \texttt{[]} && (\text{def. of } \texttt{map}) \\ &= \texttt{map}\ (g \circ f)\ \texttt{[]} && (\text{def. of } \texttt{map}) \end{aligned}$$

Step Case ($xs = z :: zs$). The IH is $P(zs) = \big(\texttt{map}\ g\ (\texttt{map}\ f\ zs) = \texttt{map}\ (g \circ f)\ zs\big)$. $P(z :: zs)$ is shown by the derivation:

$$\begin{aligned} \texttt{map}\ g\ (\texttt{map}\ f\ (z :: zs)) &= \texttt{map}\ g\ (f\ z :: \texttt{map}\ f\ zs) && (\text{def. of } \texttt{map}) \\ &= g(f\ z) :: \texttt{map}\ g\ (\texttt{map}\ f\ zs) && (\text{def. of } \texttt{map}) \\ &= (g \circ f)\ z :: \texttt{map}\ g\ (\texttt{map}\ f\ zs) && (\text{def. of } \circ) \\ &\stackrel{\text{IH}}{=} (g \circ f)\ z :: \texttt{map}\ (g \circ f)\ zs \\ &= \texttt{map}\ (g \circ f)\ (z :: zs) && (\text{def. of } \texttt{map}) \end{aligned}$$

[9] **3.** Let e denote the CoreML expression

$$\textbf{let } g = \mathsf{Y}\ (\lambda f.\mathsf{cons}\ 0\ f)\ \textbf{in } g$$

Use the environment

$$E = \{\mathsf{Y} : (\alpha_0 \to \alpha_0) \to \alpha_0, \mathsf{cons} : \alpha_1 \to \mathsf{list}(\alpha_1) \to \mathsf{list}(\alpha_1), 0 : \mathsf{int}\}$$

to transform the type inference problem $E \rhd e : \alpha_2$ into a unification problem.

Solution.

$$E \rhd e : \alpha_2$$

$$\stackrel{\mathsf{let}}{\Rightarrow}$$

$$E \rhd \mathsf{Y}\ (\lambda f.\mathsf{cons}\ 0\ f) : \alpha_3; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{app}}{\Rightarrow}$$

$$E \rhd \mathsf{Y} : \alpha_4 \to \alpha_3; E \rhd \lambda f.\mathsf{cons}\ 0\ f : \alpha_4; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{con}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; E \rhd \lambda f.\mathsf{cons}\ 0\ f : \alpha_4; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{abs}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; E, f : \alpha_5 \rhd \mathsf{cons}\ 0\ f : \alpha_6; \alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{app}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; E, f : \alpha_5 \rhd \mathsf{cons}\ 0 : \alpha_7 \to \alpha_6; E, f : \alpha_5 \rhd f : \alpha_7;$$
$$\alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{app}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; E, f : \alpha_5 \rhd \mathsf{cons} : \alpha_8 \to \alpha_7 \to \alpha_6; E, f : \alpha_5 \rhd 0 : \alpha_8;$$
$$E, f : \alpha_5 \rhd f : \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{con}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; \alpha_1 \to \mathsf{list}(\alpha_1) \to \mathsf{list}(\alpha_1) \approx \alpha_8 \to \alpha_7 \to \alpha_6; E, f : \alpha_5 \rhd 0 : \alpha_8;$$
$$E, f : \alpha_5 \rhd f : \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{con}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; \alpha_1 \to \mathsf{list}(\alpha_1) \to \mathsf{list}(\alpha_1) \approx \alpha_8 \to \alpha_7 \to \alpha_6; \mathsf{int} \approx \alpha_8;$$
$$E, f : \alpha_5 \rhd f : \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{con}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; \alpha_1 \to \mathsf{list}(\alpha_1) \to \mathsf{list}(\alpha_1) \approx \alpha_8 \to \alpha_7 \to \alpha_6; \mathsf{int} \approx \alpha_8;$$
$$\alpha_5 \approx \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6; E, g : \alpha_3 \rhd g : \alpha_2$$

$$\stackrel{\mathsf{con}}{\Rightarrow}$$

$$(\alpha_0 \to \alpha_0) \to \alpha_0 \approx \alpha_4 \to \alpha_3; \alpha_1 \to \mathsf{list}(\alpha_1) \to \mathsf{list}(\alpha_1) \approx \alpha_8 \to \alpha_7 \to \alpha_6; \mathsf{int} \approx \alpha_8;$$
$$\alpha_5 \approx \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6; \alpha_3 \approx \alpha_2$$