# Game Theory — Standard Auctions

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## Topics

Sufficient time and interest provided, we'll cover

a. General
b. Revelation principle
c. Myerson-Satterthwaite Theorem
d. First price (& Dutch) Auction
e. Second price (& English) Auction
f. All-pay Auction.

If these topics drive you mad with desire for more, I suggest reading

- Krishna, V., "Auction Theory," Academic Press, 2002,
- Milgrom, P., "Putting Auction Theory to Work," CUP, 2004.

# Things sold through auctions

Radio spectrum, electricity, prize bulls, race horses, off-shore oil leases, timber, antiques, tobacco, fresh fish, used cars, cut flowers (Holland), coins, stamps, wine, art, houses (80% in Australia), U.S. Treasury bills, search results, stocks, pork bellies, chicken (contracts), gold, wheat, foreign currency, TV broadcast rights for sports, corporations, celebrities' signatures, slices of toasts bearing the face of Jesus, fake U2 concert tickets, last month's Lehman credit swap auction...





# General

What if sellers only have statistical information about the preferences (valuations) of their potential customers?

This is a re-interpretation of the Monopolists' problem from general micro theory ($MR = MC$). Which I assume you don't know. The thing is, it only works for perfect information.

The design problem of finding the **optimal** (ie revenue maximising) mechanism for any number of objects and preferences is not solved. We focus on **efficiency**.

The subject concerned with finding such mechanisms is called **mechanism design**. It got last year's Nobel prize in economics.

# Auctions are not really a new idea

- Used by the Babylonians (500 bc).
- First Roman fire brigade offered to buy the burning house and only extinguished the fire if the offer was accepted.
- After having killed Emperor Pertinax, the Prætorian Guard auctioned off the Roman Empire (193 ad).
- Johann Wolfgang von Goethe sold a manuscript through a second-price auction (1797).
- The biggest revenue yet was generated by the US FCC spectrum auctions (1994–2008).

So there are established computer science auctions courses at: MIT, Harvard, Carnegie Mellon, Stanford &c. There are in-house courses at Microsoft, Google and eBay.





# Typical auction objectives

- Efficiency.
- Revenue maximisation.
- Information aggregation and revelation.
- Valuation and price discovery.
- Transparency and fairness.
- Speed and low admin cost.
- Fostering competition (entry).

# Single-object standard auctions

Assume you only want to auction a single good. If we are further prepared to assume that the potential buyers' have private valuations which are independently distributed, then there is a set of simple yet powerful results for the 4 standard auctions:

1. First price auction
2. Dutch auction
3. Second price auction
4. English auction.

Our main criterion to analyse these auction types is efficiency.

**Def.** We call a mechanism ex-post efficient, if it allocates the object to the player with the highest valuation.





# Assumptions

In what we'll do today, we'll

- look at independent private values (well relax this),
- ignore risk (can be relaxed),
- are looking for a symmetric equilibrium (can be relaxed).

The hard assumption is, that

- bidders with higher valuations submit higher bids. (If this fails, we can go home now.)

# Probability distributions

The probability distribution of a real-valued random variable $\mathcal{X}$ is completely characterised by its **cumulative distribution fn** (cdf)

$$F(x) = \Pr(\mathcal{X} \leq x), \ \forall x \in \mathbb{R}.$$

A **probability distribution fn** (pdf) $f(\cdot)$ describes the range of possible values that some random variable $\mathcal{X}$ can attain and the probability that the value of the random variable is within any subset of that range. For real-valued, continuous random variables (assuming differentiability of $F$), it is given by

$$f(x) = \frac{\partial F(x)}{\partial x}, \ \forall x \in \mathbb{R}.$$

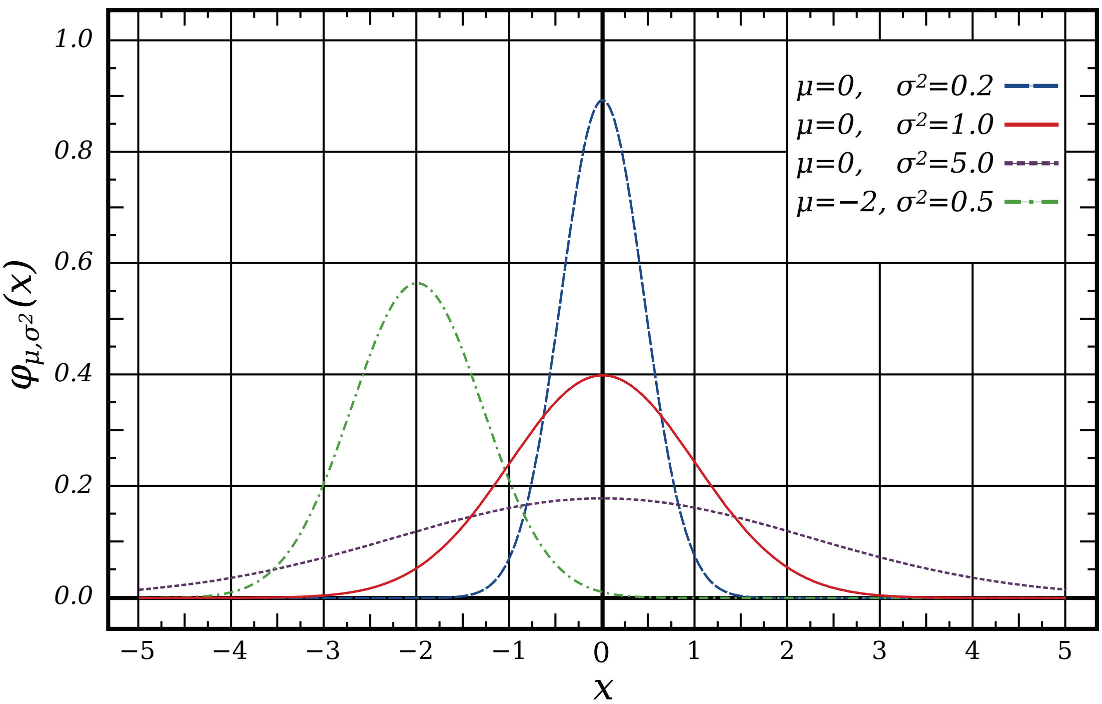
Discrete distributions are characterised by a probability mass fn

$$p(x) = \Pr(\mathcal{X} = x), \ \forall x \in \mathbb{R}.$$

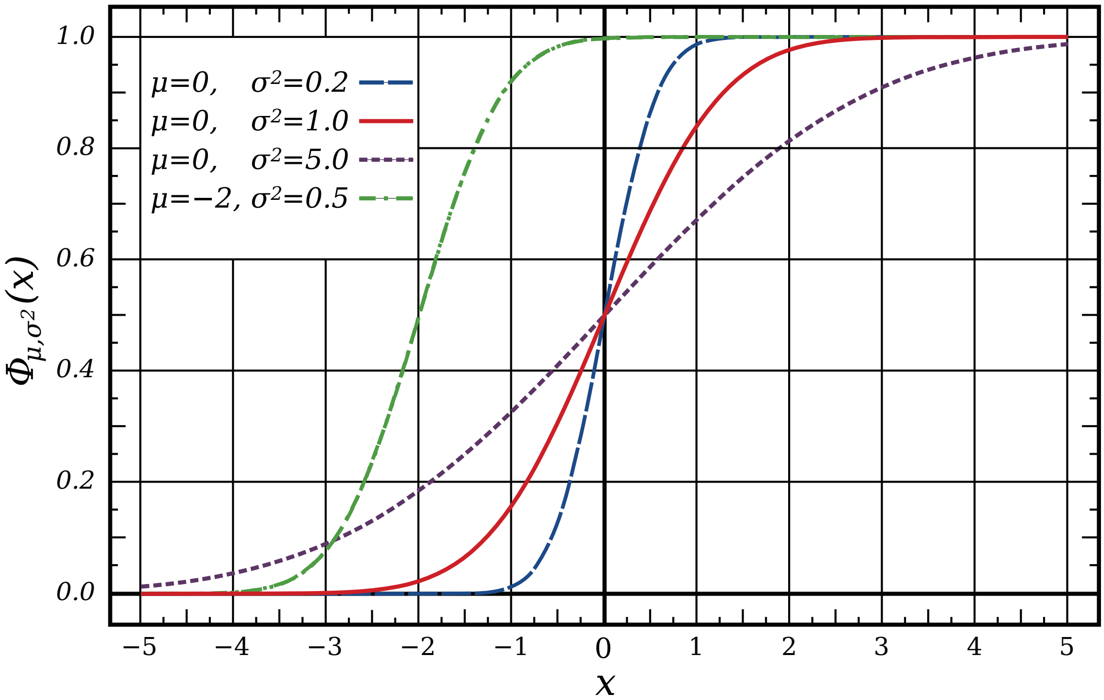




# Normal pdf: $f(x) = \varphi_{\mu,\sigma^2}(\mathcal{X})$



# Normal cdf: $F(x) = \Phi_{\mu,\sigma^2}(\mathcal{X})$








# Probability distributions

Pdf's are useful if we need to form expectations of the realisation of a random variables, ie take the sum over its possible realisations. Its expectation is

$$\int_{-\infty}^{+\infty} xf(x)dx = \mu.$$

The cdf $F(x)$ is generated by

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

Throughout this lecture we will use the Uniform distribution $U_{[0,1]}$ which assigns the same probability to each possible realisation.

So all pdfs, cdfs we see today will be trivial—but they don't have to be.

# Independent private values (IPV) framework

- A seller auctions a single object which he values at $\bar{u} = 0$.
- There are $N$ risk-neutral potential buyers. Their valuations for the object $v_i$, $i = \{1, \ldots, N\}$ are drawn independently from the (same) interval $V_i \equiv [0, 1]$. Valuations follow the distribution $F(v_i)$ with atomless probability density $f(v_i)$.
- Since the bidders' valuations are all drawn from the same distribution $F_{[0,1]}$, bidders are called ex-ante symmetric.
- Since each bidder $i$ only knows his own valuation $v_i$, these valuations are called private values. A players' private information is called his Type.
- Player $i$ only know the distributions $F_{[0,1]}$ of their competitors $(-i)$ valuations. Similarly, the seller only knows the distribution of the potential buyers' values $F_{[0,1]}$.





# First price auction

**Rules:** Each bidder submits a single, sealed bid to the seller. The seller gives the object to the highest bidder who pays his own bid.

- A players' optimal bidding strategy depends on her opponents' offers — which she cannot observe!
- How should a player act in such a game of incomplete information?
- Bidder $i$'s strategy specifies a bid for each of his possible valuations of the object. This strategy is called a bidding function $b_i : [0, 1] \mapsto \mathbb{R}_+$.
- Hence we are looking for a (symmetric) Bayes-Nash equilibrium (BNEq'm) of these bidding functions.

# Assumptions

We do not know too much about these bidding functions $b_i(v_i)$. But the following assumptions seem (sort of) natural.

**Assumption 1:** Bidders with higher valuations make higher bids. Hence $b_i(v_i)$ is strictly monotonic for all bidders $i$.

**Assumption 2:** Ex-ante symmetric bidders submit identical bids for identical valuations: $b_i(v_i) = b(v_i)$, $\forall i$.

Assumption 2 implies that we are looking for symmetric BNEq'a.

The final assumption is made for expositional simplicity. Contrary to the above two assumptions it is not at all essential.

**Assumption 3:** Valuations uniformly distributed: $v_i \sim U_{[0,1]}$ for all bidders $i$.



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# Bayes-Nash Equilibrium (BNEq'm)

Let's define a symmetric BNEq'm: It is a profile of bids by all players $b^* = (b_1^*, \ldots, b_N^*)$ such that, for each player $i \in N$ with valuation $v_i \in V$, it is true that

$$b_i^* \in \underset{b_i \in [0,\infty)}{\operatorname{argmax}} \int_0^1 \underbrace{\int_0^1 \cdots \int_0^1}_{v_{-i} \in V_{i-1}} f(v) u_i(b_i, b(v_{-i})^*) \, dv_i \, dv_{-i}.$$

where $v = v_1, \ldots, v_N$, and $v_{-i} = v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N$.

Since players have private information (their value), they must form 'beliefs' about their opponents' valuations. These are independent and identical (iid) under the assumptions made. Moreover, they equal the commonly known ex-ante probabilities given by $f(v_i)$.

# Digression 1

**Revelation principle**





# Revelation principle

A real bidding strategy may be a complicated object. It may depend on many things such as the weather, whether I met someone interesting yesterday, or the twists of a divining rod. Hence we have to examine a very large set of potential mechanisms (each based on a different specification of bidding strategies) in order to find the efficient or optimal mechanism.

As it turns out, the following result says that we can restrict the investigation to 'simple' mechanisms with bids of the form $b(v_i)$.

For each BNEq'm of a game of incomplete information there is a related game in which strategies only depend on the players' types ('direct revelation game'). In this game, it is a BNEq'm for the players to 'announce' their true type. Moreover, the game has the same payoffs as the original game.

# A delegation game

Assume we found a BNEq'm $b(v)$.

- By definition, $b(v_i)$ is payoff maximising for player $i$ with type $v_i$, given that the competitors bid $b(v_{-i})$ (using their symmetric bidding fns).
- Say, bidder $i$ cannot personally participate in the auction but sends a representative. This representative knows (and employs) the symmetric equilibrium bidding fn $b(\cdot)$ but does not know player $i$'s type $v_i$.
- Now $i$'s representative calls player $i$ during the auction and asks for the argument of his bidding function. What will player $i$ announce?

Of course $v_i$, his <u>true</u> valuation!





# Digression 2

**Efficient trade:**
**Myerson-Satterthwaite Theorem**

# Efficient trade

- There are just two players: a seller and a buyer.
- Private buyer valuation $v \sim U_{[0,1]}$.
- Private seller cost $c \sim U_{[0,1]}$.
- There are no outside subsidies.
- It is ex-post efficient to trade whenever $v > c$.
- We are looking for any mechanism which implements voluntary trade whenever $v > c$.





# Efficient mechanism

- The mechanism
  - asks buyer & seller for their types $v$ & $c$,
  - charges $p(v, c)$ from the buyer,
  - and pays $p(v, c)$ to the seller.
- If there is efficient trade, then the expected gains from trade are

$$\int_0^1 \int_0^v (v - c)\, dc\, dv = \int_0^1 \frac{v^2}{2} dv = \frac{1}{6}.$$

# Buyer

- No-one forces the buyer to participate in our mechanism and report his valuation truthfully. Hence voluntarily reporting his <u>true</u> valuation $v$ must be optimal for him.
- Consider the general case where a buyer of valuation $v$ (mis-)reports $\hat{v}$ to the mechanism.
- Notice that a buyer who announces $\hat{v}$ gets the object with probability $\hat{v}$ under the assumed uniform dis'n.
- The utility a buyer's of type $v$ has from reporting $\hat{v}$ is

$$\text{argmax}_{\hat{v}}\, u(\hat{v}, v) = v\hat{v} - \mathbb{E}_c\, p(\hat{v}, c).$$

- If truth-telling is optimal to the buyer, then this utility must be maximised at $\hat{v} = v$:

$$\frac{d}{dv}u(\hat{v}, v) = \underbrace{\frac{\partial}{\partial \hat{v}}u(\hat{v}, v)}_{\text{foc}=0} + \frac{\partial}{\partial v}u(\hat{v}, v) = \frac{\partial}{\partial v}u(\hat{v}, v) = \hat{v}\Big|_{\hat{v}=v} = v.$$





# Buyer

- Integrating up the expected surplus for all types of buyers gives the average buyers' surplus as (we use integration by parts with $f = u(v, v), g' = 1, g = (v-1)$)

$$\begin{aligned} \int_0^1 u(v, v)dv &= (v-1)u(v, v)\Big|_{v=0}^{1} - \int_0^1 (v-1)\frac{d}{dv}u(v, v)\, dv \\ &= \int_0^1 (1-v)v\, dv = \frac{1}{6}. \end{aligned}$$

(Notice: A type $v = 0$ buyer should get utility 0.)

- Our result is that for a buyer to report truthfully, he must get <u>all</u> gains from trade which exist between the two parties.

# Seller

- But the problem for the buyer is completely symmetric! She must get <u>all</u> gains from trade which exist between the two parties as well in order to report her cost truthfully.
- Since both is impossible at the same time, we showed that it is impossible to find a mechanism which implements efficient trade with private information!
- Intuition: Only if all surplus 'belongs' to a single party, she has the right incentives to maximise the value of this surplus.





# Myerson-Satterthwaite Theorem

- This remarkably strong result comes by the name of Myerson-Satterthwaite Theorem.
- More generally, it says that efficient trade is impossible with private information drawn from overlapping type spaces.
- Hence, buyers tend to shield their bids by offering less than their true valuation.
- Similarly, sellers tend to specify higher costs than they actually incur.
- The resulting inefficiency is muted in (very) large trading environments but can be severe in 1:1 negotiations.

# First price auction

**Back to the first price auction.**
**(We will not consider strategic sellers.)**





# Maximisation problem

As for any other NEq'm, a condition for BNEq'm is that bids are mutual best responses.

Formally, a strategy profile $b(v)$ constitutes a BNEq'm of the first price auction (FPA) if, for each value $v_i$, $b(v_i)$ satisfies

$$\underset{b_i}{\operatorname{argmax}}\,(v_i - b_i)\,\mathrm{pr}(b_i \geq b(v_j))_{\forall j \neq i}.$$

This is not a simple problem to solve. We simplify the analysis by looking for linear and symmetric bidding functions $b(v_i) = \alpha + \gamma v_i$. (If we indeed find such a linear fn, then this is no assumption!)

We notice the independent draw of two (or more) identical valuations is a probability zero event for atomless distributions. Hence the problem reduces to

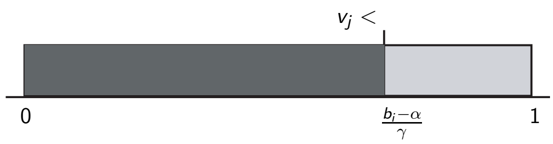
$$\underset{b_i}{\operatorname{argmax}}\,(v_i - b_i)\,\mathrm{pr}(b_i > \alpha + \gamma v_j)_{\forall j \neq i}.$$

# Probabilities

The probability of issuing a higher bid than that of player $j \neq i$ is

$$\text{pr}(b_i > \alpha + \gamma v_j) = \text{pr}\left(v_j < \frac{b_i - \alpha}{\gamma}\right)$$

which equals under the uniform




$$\text{pr}\left(v_j < \frac{b_i - \alpha}{\gamma}\right) = \frac{b_i - \alpha}{\gamma}.$$

Since we consider $N-1$ independently drawn valuations, we get

$$\text{pr}(b_i > \alpha + \gamma v_j)_{\forall j \neq i} = \underbrace{\frac{b_i - \alpha}{\gamma}\frac{b_i - \alpha}{\gamma}\cdots\frac{b_i - \alpha}{\gamma}}_{\text{for all } j \neq i \text{, ie } N-1 \text{ times}} = \left(\frac{b_i - \alpha}{\gamma}\right)^{N-1}.$$





Let's assume with Nash that all opponents of $i$ use their eq'm strategy $b(v_j) = \alpha + \gamma v_j$ in equilibrium.

What is bidder $i$'s best response to these $b(v_j)$?

$$\underset{b_i}{\text{argmax}}\ u(v_i, b_i) = (v_i - b_i)\left(\frac{b_i - \alpha}{\gamma}\right)^{N-1}.$$

Bidder $i$ chooses his optimal bid $b_i$ by maximising his utility

$$\frac{\partial u(v_i, b_i)}{\partial b_i} = 0: \quad (N-1)(v_i - b_i)\left(\frac{b_i - \alpha}{\gamma}\right)^{N-2}\frac{1}{\gamma} - \left(\frac{b_i - \alpha}{\gamma}\right)^{N-1} = 0.$$

Since a bidder of type 0 will bid 0, we know that $\alpha = 0$. Hence

$$b_i = \frac{v_i(N-1) - \alpha}{N} = v_i\frac{N-1}{N} = b^*(v_i).$$

So we found a linear eq'm bidding fn. This justifies our tentative assumption we made earlier.

# Revenue

Which revenue is generated for the seller by the FPA?

With types uniformly distributed on $[0, 1]$, the seller's revenue is given by

$$\begin{aligned}\mathbb{E}[\Pi_F] &= \int_0^1 b(v)\ \text{pr(highest type)}\ dv \\ &= \int_0^1 \frac{n-1}{n} vnv^{N-1} dv = \frac{n-1}{n+1}.\end{aligned}$$





# Discussion

It can be shown that the equilibrium we just derived is the only symmetric equilibrium of the FPA.

Since

$$\frac{\partial b^*(v_i)}{\partial vi} = \frac{\partial}{\partial vi}\left(v_i \frac{N-1}{N}\right) = \frac{N-1}{N} > 0,$$

we know that the bidding fn is increasing. Hence the object is always going to the bidder with the highest valuation (see Assumption 1). Ie, the FPA is ex-post <u>efficient</u>.

However, a participant in the FPA will <u>not</u> report his true valuation but something lower. This lower bid is the expected second highest bid (given winning).

This is actually quite intuitive: The optimal winning offer should be just a little higher than the second highest bid.

# Dutch auction

**Rules:** The seller starts by indicating a high price and lowers it continuously. The first bidder signalling acceptance of the current price gets the object at the current price.

**Theorem.** With independent private values, it is a BNEq'm of the Dutch auction to signal acceptance as soon as the price reaches

$$v_i \frac{N-1}{N}.$$

**Proof.** The bidders' optimisation problem is identical to that in the FPA.





# Second price auction

**Rules:** Each bidder submits a sealed bid to the seller. The seller gives the object to the highest bidder who pays the second highest bid.

To save on notation, we consider a 2-player variant of the same IPV-model as for the FPA. In particular, we still make assumptions 1–3.

Bidder $i$'s maximisation problem in the second price auction (SPA) is, for values $v = v_1, \ldots, v_N$, strictly monotonic equilibrium bidding fn $b(v_j)$ and $j = 3 - i$ given by

$$\underset{b_i}{\operatorname{argmax}} \, (v_i - b(v_j)) \, \mathrm{pr}(b_i \geq b(v_j)).$$

Bidder $i$'s expected utility when of type $v_i$ and bidding $\hat{b}$ is

$$u_i(v_i, \hat{b}) = \underbrace{\int_0^{\hat{v}_i = b^{-1}(\hat{b})} (v_i - b(v_j))\, f(v_j) dv_j}_{v_j \leq \hat{v}_i} .$$

Takeing derivatives wrt $\hat{b}$ (employing Leibnitz' rule) gives the foc

$$\left( v_i - \underbrace{b(b^{-1}(\hat{b}))}_{=\hat{b}} \right) \frac{db^{-1}(\hat{b})}{d\hat{b}} \bigg|_{v_i = b^{-1}(\hat{b})} = 0$$

or

$$v_i = \hat{b} = b^*(v_i).$$

It is a BNEq'm to bid one's true valuation!



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General Revelation principle Myerson-Satterthwaite First price auction **Second price auction** All-pay auction

# Weakly dominant strategies

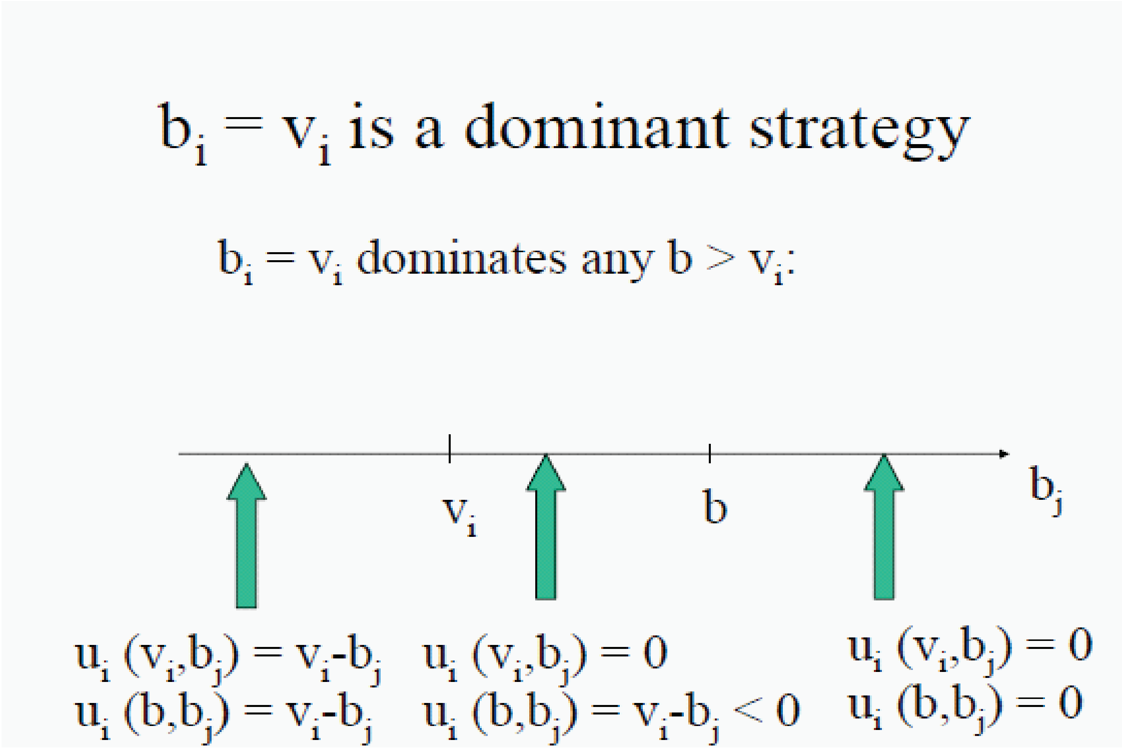
We now show that the result we just derived is much stronger than BNEq'm: it holds in (weakly) dominant strategies. This is nice since dominance implies that the chosen strategies are optimal independently of what the opponents do. In particular, it is independent of the players' 'beliefs' about their opponents (ie the value distributions).

**Theorem.** It is a weakly dominant strategy to bid one's true valuation, ie $b_i^* = v_i$, in the SPA.

**Proof.** As before we restrict attention to the two players case, ie $i = 1, 2$ and $j = 3 - i$.

Notice: Should player $i$ win, then his bid $\hat{b}_i$ has no influence on the price he pays; the price is exclusively determined by $b_j = b(v_j)$. The own offer $\hat{b}_i$ 'only' determines whether $i$ wins or not.

# Proof








# Proof

1. Let $b_j < v_i = b_i^*$
   - if $\hat{b}_i > b_j$, then both bids give the same payoff.
   - If $\hat{b}_i \leq b_j$, then $b_i^* = v_i$ is <u>better</u> than $\hat{b}_i \neq v_i$ since it gives a higher payoff.
2. Let $b_j > v_i = b_i^*$
   - if $\hat{b}_i \geq b_j$, then $b_i^* = v_i$ is <u>better</u> than $\hat{b}_i \neq v_i$ since it avoids a loss,
   - if $\hat{b}_i < b_j$, then both bids give the same payoff.
3. Let $b_j = v_i = b_i^*$
   - if $\hat{b}_i > b_j$, then $b_i^* = v_i$ is <u>better</u> than $b_i \neq v_i$ since it avoids a loss,
   - if $\hat{b}_i \leq b_j$, then both bids give the same payoff..

In all possible cases, $b_i^* = v_i$ gives <u>at least</u> the same payoff as a deviation. Hence $b_i^*$ is weakly dominant.

# Revenue

Which revenue is generated by the SPA for the seller?

With uniformly distributed valuations, the revenue equals

$$
\begin{aligned}
\mathbb{E}[\Pi_S] &= \int_0^1 b(v) \ \text{pr(second highest type)} \ dv \\
&= \int_0^1 n(n-1)(v^{n-1} - v^n) dv = n(n-1) \left[ \frac{v^n}{n} - \frac{v^{n+1}}{n+1} \right]_0^1 \\
&= n(n-1) \frac{n+1-n}{n(n+1)} = \frac{n-1}{n+1}.
\end{aligned}
$$

To avoid didactic complications, we take pr(second highest type) from one of the literature references on page 45.





# Discussion

Under the derived bidding function $b = v_i$, the object is always going to the bidder with the highest valuation (see Assumption 1). Hence the SPA is ex-post efficient.

Since bidders announce their <u>true</u> valuations in the SPA, we found a ‘direct’ mechanism which induces the agents to voluntarily report their true private information. This had <u>enormous</u> impact on economic theory.

The intuitive reason for this result and its generalisations (Vickrey-Clark-Groves or VCG mechanisms) is that the price the winner has to pay does not depend on his bid: it is solely determined by the other players.

# English auction

**Rules:** The seller starts from a price of zero and continuously raises this price. At a zero price, all bidders participate but they can drop out whenever they want. (There is no re-entry.) The auction is over when the penultimate bidder leaves the auction. The remaining bidder wins and pays the price at which the last opponent dropped out.

**Theorem.** With independent private values, it is a BNEq'm of the English auction to leave the auction as soon as the own value is reached.

**Proof.** The bidders' optimisation problem is identical to that in the SPA.





# Correlated valuations: The winner's curse

*"I paid too much for it, but it's worth it."* (Sam Goldwyn)

- We now relax the 'I' among the IPV assumptions.
- Consider 'common' valuations like $\hat{v}_1 = \dfrac{v_1 + v_2}{2}$.
- The winner's curse is the fact that the bidder who most overestimates the value of the object wins the bidding. This may not be good news.
- Strategically, a bidder should reduce his bid to adjust for this effect. And more so, the more bidders there are.
- Uncertain resale value can induce the winner's curse. Oil wells, stock markets, or arts auctions are another examples.

# All-pay auction

**Rule:** Each bidder submits a single, sealed bid to the seller. The seller gives the object to the highest bidder. Everyone pays their bids.

**Application:** War of attrition like scenarios as, for instance, innovation or patent races, r&d contests, or lobbying &c.

**Assumptions:** We use the same IPV model as introduced for the FPA. In particular, we still make assumptions 1–3.





# All-Pay Auction: Eq'm-check

Initially, all types $v_i$, $i = 1, \ldots, N$ are drawn independently. Assuming the existence of a strictly monotonic bidding fn $b(v_i)$, the highest bid is made by the highest type.

**Eq'm-check**: Given that all opponents of player $i$ bid their equilibrium strategy $b(v_{-i})$, is it profitable for player $i$ to deviate from the prescribed equilibrium bid $b(v_i)$ to some other bid $\hat{b}$?

More formally, $i$'s maximisation problem from bidding $\hat{b}$ is

$$\max_{\hat{b}} \; v_i \operatorname{pr}(\hat{b} > b(v_j))_{\forall j \neq i} - \hat{b}.$$

Since $b(v_i)$ is invertible ($b' > 0!$), this is equivalent to

$$\max_{\hat{b}} \; v_i \operatorname{pr}(v_j < b^{-1}(\hat{b}))_{\forall j \neq i} - \hat{b}.$$

# Order statistics

How do we determine $\mathrm{pr}(v_j < b^{-1}(\hat{b}))$ for independently drawn $v_i$, $i = 1, \ldots, N$? Notice that these probabilities only depend on the type distributions $F$!

I suggest we simply take them from one of these books:

- Krishna, V., "Auction Theory," App C, Academic Press, 2002,
- David, H. and H. Nagaraja, "Order statistics," Wiley, 2003.

**Definition.** Let $X := X_1, X_2, \ldots, X_n$ be $n$ independent draws from the same probability distribution $F(x)$ with differentiable density $f(x)$. Then the ordered random variables $Y_{(1:n)} \geq Y_{(2:n)} \geq \ldots \geq Y_{(n:n)}$ are called order statistics of the random variable $X_1, X_2, \ldots, X_n$.

Hence $Y_{(k:n)}$ is the $k$-highest random variable among random $X_1, X_2, \ldots, X_n$.





# Winning probabilities

Under the uniform these probabilities are easy. We will now employ a new and more general solution strategy which we could also have used for solving the FPA.
It is generally true that

$$\begin{aligned} F_{(1:n)}(y) &= \Pr(\max X \leq y) = \Pr(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y) \\ &= F_1(y)\ F_2(y) \cdots F_n(y) \\ &= (F(y))^n. \end{aligned}$$

Hence the probability that the types of all $N-1$ opponents are below $b^{-1}(\hat{b})$ equals

$$\mathrm{pr}(v_j < b^{-1}(\hat{b})) = (F(b^{-1}(\hat{b})))^{N-1} = (b^{-1}(\hat{b}))^{N-1}$$

(only the last step uses the Uniform).

# Symmetric Eq'm

hence player $i$'s maximisation problem simplifies to

$$\max_{\hat{b}} v_i(b^{-1}(\hat{b}))^{N-1} - \hat{b}$$

with foc

$$\frac{\partial u_i(\hat{b}, v_i)}{\partial \hat{b}} = 0 : \ (N-1)v_i \underbrace{(b^{-1}(\hat{b}))}_{=v_i^{N-2}}{}^{N-2} \underbrace{\frac{db^{-1}(\hat{b})}{d\hat{b}}}_{=1/b'(v_i)}\Bigg|_{v_i=b^{-1}(\hat{b})} = 1.$$

In symmetric equilibrium we must have that $\hat{b} = b(v_i)$ or $b^{-1}(\hat{b}) = v_i$ an thus

$$(N-1)v_i v_i^{N-2} = b'(v_i).$$



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# The bidding function

We can solve this differential equation by simply integrating up

$$(N-1)\int_0^{v_i} \tilde{v}^{N-1} d\tilde{v}_i = \int_0^{v_i} b'(\tilde{v})d\tilde{v} = b(v_i)$$

and obtain

$$(N-1)\frac{1}{n}\tilde{v}^n\Big|_0^{v_i} = \frac{n-1}{n}v_i^n = b^*(v_i).$$

The expected revenue from this auction type equals (under the uniform)

$$\mathbb{E}[\Pi_A] = n\int_0^1 b(v)dv = n\int_0^1 \frac{n-1}{n}v_i^n dv = \frac{n-1}{n+1}.$$

This is the same revenue that we derived for all other auctions (FPA, SPA and equivalents). Hence all four standard auctions and the all-pay auction give exactly the same profits to the seller.

# Why?

You can answer this question (it's called the Revenue Equivalence Theorem) by either looking it up in the references, or by taking an auctions / mechanism design course in the econ department.

I didn't spoil all fun, promise—after all, we only talked about single object auctions.

Thank you for your attention!