Game Theory

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Organisation

Time and Place

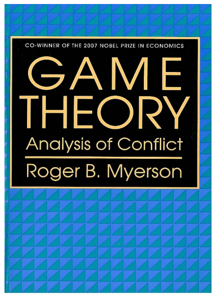
- Thursday, 16:00-18:00, HS 10

week 1	October 2	week 8	November 20
week 2	October 9	week 9	November 27
week 3	October 16	week 10	December 4
week 4	October 23	week 11	December 11
week 5	October 30	week 12	January 8
week 6	November 6	week 13	January 15
week 7	November 13	first exam	January 22

Literature & Online Material

Literature

Roger B. Myerson
Game Theory: Analysis of Conflict
Harvard University Press, 1991
ISBN: 0-674-34116-3



Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay V. Vazirani (ed.)
Algorithmic Game Theory
Cambridge University Press, 2007
ISBN 978-0-521-87282-9

Online Material & Exam

Online Material

Transparencies and homework will be available from IP starting with 138.232 after the lecture; exercises and solutions will be discussed during the lecture

Homework & Exam

- officially there are no exercises as this course is labelled VO
- however, without exercises you'll not learn anything
- I'll give bi-weekly exercises which will be discussed in the lecture

Any protest?

Schedule

week 1	motivation, introduction to decision theory
week 2	decision theory, basic model of game theory
week 3	dominated strategies, bayesian games
week 4	equilibria of strategic-form games
week 5	evolution, resistance, and risk dominance
week 6	sequential equilibria of extensive-form games
week 7	subgame-perfect equilibria
week 8	complexity of finding Nash equilibria
week 9	equilibrium computation for two-player games
week 10	refinements of equilibrium in strategic form
week 11	persistent equilibria
week 12	games with communication
week 13	sender-receiver games

Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games

equlibria of strategic-form games, evolution, resistance, and risk dominance, sequential equilibra of extensive-form games, subgame-perfect equilibra, complexity of finding Nash equilibria, equilibrium computation for two-player games

refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

What is game theory?

game theory is conceivable as the study of mathematical models of conflict and cooperation between intelligent and rational decision-makers

Why is this part of a computer science major?

- why not, it is part of computer science major at Georgia Tech
- why not, Ariel Rubinstein writes

  > *There are many similarities between logic and game theory. Whereas logic is the study of truth and inference, game theory is the study of strategic considerations.*

  and Scott Shenker writes

  > *the Internet is the equilibrium, we just have to identify the game*

History of game theory

Zermelo	*Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*	1913	✓
Borel	*La Théorie du Jeu et les Équations Intègrales à Noyau Symètrique*	1921	✓
von Neumann	*Zur Theorie der Gesellschaftsspiele*	1928	✓
von Neumann, Morgenstern	*Theory of Games and Economic Behaviour*	1944	✓
Nash	*The Bargaining Problem*	1950	
Harsanyi	*A Simplified Bargaining Model for the n-Person Cooperative Game*	1963	
Selten	*Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit*	1965	

Logic and Games

Games in the history of logic

it can be argued that already Aristotle made the connection between (propositional) logic and games; syllogism are introduced and argued for in the context of a specific game, a debate

Logical Games

- these are two-player games: $\forall$belard and $\exists$loise
- the domain of the game is set Ω
- $\forall$ and $\exists$ play by choosing elements from Ω
- an infinite sequence of elements of Ω is called a play
- a finite sequence is called position
- for each position a, τ(a) decides the next player
- $W_\forall$ ($W_\exists$) denotes the set of plays where $\forall$ ($\exists$) wins

Example — Hintikka game

consider the language of first-order, the game is played on formulas

- legal moves for $\forall$:
  - given $A \wedge B$, $\forall$ chooses either A or B
  - given $\forall x F(x)$, $\forall$ chooses some instance a
- legal moves for $\exists$:
  - given $A \vee B$, $\exists$ chooses either A or B
  - given $\exists x F(x)$, $\exists$ chooses some instance a
- the game for $\neg F$ is the dual of F
- $\exists$ wins if and only if the encountered atomic formula is true

Lemma

given a formulas F, call the Hintikka game $\mathcal{G}(F)$; then $\exists$ has a winning strategy if and only if F is valid (in first-order logic)

Observation

idea can be extended to temporal logics, in particular to the μ-calculus, etc.

Content

Decision-Theoretic Foundations

Definition — game

- a game refers to any social situation involving two or more individuals
- the players are supposed to be rational and intelligent
- rational means the player makes decisions consistently in pursuit of her own objective
- intelligent means the player can make the same inferences about the game that we can make

Definition — probability distribution

let Z be a finite set, the probability distributions $\Delta(Z)$ over Z are defined as follows:

$$\Delta(Z) = \{q \colon Z \to \mathbb{R} \mid \sum_{y \in Z} q(y) = 1 \text{ and } \forall z \in Z\ q(z) \geqslant 0\}$$

Definition — lottery

- let Ω denote set of possible states, and let X denote the set of prizes
- Ω and X are finite
- a lottery is a function f that maps Ω to $\Delta(X)$ such that

$$\sum_{x \in X} f(x|t) = 1 \qquad t \in \Omega$$

- the set of lotteries is defined as follows:

$$L = \{f \mid \Omega \to \Delta(X)\}$$

- let t be a state, $f(\cdot|t)$ denotes the probability distribution over X in t:

$$f(\cdot|t) = (f(x|t))_{x \in X} \in \Delta(X)$$

Example — roulette lotteries

suppose $\Omega = \varnothing$, then lotteries depend only objective unknowns that can be assigned probabilities, for example the prize is determined by a coin toss

Example — horse lotteries

suppose the only probabilities that occur in $\Delta(X)$ are 1 or 0, then the final prize (say the share value) depends only on the state (the policy of the US), such states are called subjective unkowns

Definition — event

- an event is a subset of Ω
- the sets of all events Ξ is defined as

$$\Xi = \{S \mid S \subseteq \Omega \text{ and } S \neq \varnothing\}$$

Definition

let f, g be lotteries and S an event

- we write $f \succcurlyeq_S g$ if f is at least as desirable as g
- $f \sim_S g$ iff $f \succcurlyeq_S g$ and $g \succcurlyeq_S f$
- $f \succ_S g$ iff $f \succcurlyeq_S g$ and $f \not\preccurlyeq_S g$

Definition

let $\alpha \in [0,1]$, let f, g be lotteries

- the compound lottery $\alpha f + (1-\alpha)g$ is the defined as:

$$\alpha f + (1-\alpha)g(x|t) = \alpha f(x|t) + (1-\alpha)g(x|t)$$

- the lottery $[x]$ always get prize x for sure:

$$[x](y|t) = 1 \quad \text{if } y = x \qquad\qquad [x](y|t) = 0 \quad \text{if } y \neq x$$

  where $t \in \Omega$

Example

what is $\alpha[x] + (1-\alpha)[y]$?

Axiomatic presentation

Axiom (totality)

- $f \succcurlyeq_S g$ or $g \succcurlyeq_S f$
- if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $f \succcurlyeq_S h$

Axiom (relevance)

if for all $t \in S$: $f(\cdot|t) = g(\cdot|t)$, then $f \sim_S g$

Axiom (monotonicity)

if $f \succ_S h$ and $0 \leqslant \beta < \alpha \leqslant 1$, then $\alpha f + (1-\alpha)h \succ_S \beta f + (1-\beta)h$

Axiom (continuity)

if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $\exists\, \gamma \in [0,1]$ such that $g \sim_S \gamma f + (1-\gamma)h$

Substitution Axioms

Axiom (objective substitution)
if $e \succcurlyeq_S f$ and $g \succcurlyeq_S h$ and $\alpha \in [0,1]$, then $\alpha e + (1-\alpha)g \succcurlyeq_S \alpha f + (1-\alpha)h$

Axiom (strict objective substitution)
if $e \succ_S f$ and $g \succcurlyeq_S h$ and $\alpha \in (0,1]$, then $\alpha e + (1-\alpha)g \succ_S \alpha f + (1-\alpha)h$

Axiom (subjective substitution)
if $f \succcurlyeq_S g$ and $f \succcurlyeq_T g$ and $S \cap T = \varnothing$, then $f \succcurlyeq_{S \cup T} g$

Axiom (strict subjective substitution)
if $f \succ_S g$ and $f \succ_T g$ and $S \cap T = \varnothing$, then $f \succ_{S \cup T} g$

Regularity Axioms

Axiom (interest)
$\forall\, t \in \Omega$, $\exists\, x, y \in X$ such that $[y] \succ_{\{t\}} [x]$

Axiom (state neutrality)
$\forall\, r, t \in \Omega$, if $f(\cdot|r) = f(\cdot|t)$ and $g(\cdot|r) = g(\cdot|t)$, and $f \succcurlyeq_{\{r\}} g$, then $f \succcurlyeq_{\{t\}} g$

Definition conditional-probability
a conditional-probability function is any function $p \colon \Xi \to \Delta(\Omega)$ such that

$$p(t|S) = 0 \qquad \text{if } t \notin S \qquad\qquad \sum_{r \in S} p(r|S) = 1$$

Definition utility function
a utility function is any function from $u \colon X \times \Omega \to \mathbb{R}$

Expected Utility Maximisation Theorem

Definition
let p denote a conditional-probability function and u any utility function, then the expected utility determined by lottery f is defined as:

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in X} u(x,t) f(x|t)$$

Theorem
the axioms (with state neutrality) are satisfied if and only if there exists a (state-independent) utility function u and a conditional-probability function such that

1. $\max_{x \in X} u(x,t) = 1$ and $\min_{x \in X} u(x,t) = 0$
2. $p(R|T) = p(R|S)p(S|T)\ \forall\, R, S, T$ so that $R \subseteq S \subseteq T$ and $S \neq \varnothing$ where $p(R|S) = \sum_{r \in R} p(r|S)$
3. $f \succcurlyeq_S g$ if and only if $E_p(u(f)|S) \geqslant E_p(u(g)|S)$