# Game Theory

Georg Moser

Institute of Computer Science @ UIBK

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## Summary of Last Lecture

### Notation

- $M$ denotes the set of $m$ pure strategies of player 1 and $N$ denotes the set of $n$ pure strategies of player 2

$$M = \{1, \dots, m\} \qquad N = \{m+1, \dots, m+n\}$$

- $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{m \times n}$

### Theorem

best response

let $x$, $y$ be be mixed strategies, then $x$ is best response to $y$ if and only if

$$x_i > 0 \quad \text{implies} \quad (Ay)_i = u = \max\{(Ay)_k \mid k \in M\} \qquad \forall i \in M$$

### Proof

on blackboard

## Definition

- a polyhedron $P \in \mathbb{R}^d$ is a set
$$\{z \in \mathbb{R}^d \mid Cz \leqslant q\} \quad \text{for some matrix } C\text{, vector } q$$
- $P$ is full-dimensional if it has dimension $d$
(i.e., $d+1$ (but not more) affinely independent elements)
- $P$ is a polytope if bounded
- the face of $P$ is $\{z \in P \mid c^\top z = q_0\}$
for $c \in \mathbb{R}^d$, $q_0 \in \mathbb{R}$
- a vertex of $P$ is the unique element of a zero-dimensional face of $P$
- an edge is a one-dimensional face of $P$
- a facet of a $d$-dimensional $P$ is a $d-1$-dimensional face

## Observation

Any nonempty face $F$ of a polyhedron $P$ can be obtained by turning some of the inequalities of $P = \{z \in \mathbb{R}^d \mid Cz \leqslant q\}$ into equalities; such inequalities are called binding





# The Best Response Polyhedron

## Definition

best response polyhedra for player 1 and 2

$$\overline{P} = \{(x, v) \in \mathbb{R}^m \times \mathbb{R} \mid x \geqslant \mathbf{0}, \mathbf{1}^\top x = 1, B^\top x \leqslant \mathbf{1}v\}$$
$$\overline{Q} = \{(y, u) \in \mathbb{R}^n \times \mathbb{R} \mid Ay \leqslant \mathbf{1}u, y \geqslant \mathbf{0}, \mathbf{1}^\top y = 1\}$$

## Example

consider $\Gamma$

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}$$

then

$$\overline{Q} = \left\{ (y_4, y_5, u) \;\middle|\; \begin{array}{l} 3y_4 + 3y_5 \leqslant u,\ 3y_4 + 5y_5 \leqslant u,\ 6y_5 \leqslant u, \\ y_4 \geqslant 0,\ y_5 \geqslant 0,\ y_4 + y_5 = 1 \end{array} \right\}$$

### Definition

a point $(y, u) \in \overline{Q}$ has label $k \in M \cup N$ if

- the $k^{th}$ inequality in the definition of $\overline{Q}$ is binding
- i.e., $\sum_{j \in N} a_{kj} y_j = u$ if $k = i \in M$ or
- for $k = j \in N$, $y_j = 0$

### Example

the point $(\frac{2}{3}, \frac{1}{3}, 3)$ has labels 1 and 2, as $x_1$, $x_2$ are best responses to $y$ for player 1 that yields pay-off 3

### Lemma

an equilibrium $(x, y)$ is a pair such that

- pair $((x, v), (y, u)) \in \overline{P} \times \overline{Q}$
- this pair is completely labeled, i,e.
  every label $k \in M \cup N$ labels either $(x, v)$ or $(y, u)$





## Equilibria by Vertex Enumeration

### Assumptions

suppose $A$ and $B^\top$ are non-negative and have no zero columns

### Algorithm

- INPUT: a nondegenerate bimatrix game
- OUTPUT: all Nash equilibria

### Method

1. $\forall\ x \in P \setminus \{\mathbf{0}\}$
2. $\forall\ y \in Q \setminus \{\mathbf{0}\}$
3. if $(x, y)$ is completely labeled, output the Nash equilibrium

$$(x \cdot \frac{1}{\mathbf{1}^\top x}, y \cdot \frac{1}{\mathbf{1}^\top y})$$

# Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games

equilibria of strategic-form games, evolution, resistance, and risk dominance, two-person zero-sum games

efficient computation of Nash equilibria

sequential equilibria of extensive-form games, subgame-perfect equilibria, complexity of finding Nash equilibria, equilibrium computation for two-player games





consider $\Gamma$, $A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}$

the polyhedra $\overline{P}$, $\overline{Q}$ are defined as follows:

$$\overline{P} = \left\{ (x_1, x_2, x_3, v) \mid \begin{array}{ll} x_1 \geqslant 0 & ① \\ x_2 \geqslant 0 & ② \\ x_3 \geqslant 0 & ③ \\ 3x_1 + 2x_2 + 3x_3 \leqslant v & ④ \\ 2x_1 + 6x_2 + 1x_3 \leqslant v & ⑤ \\ x_1 + x_2 + x_3 = 1 & \end{array} \right\}$$

$$\overline{Q} = \left\{ (y_4, y_5, u) \mid \begin{array}{ll} 3y_4 + 3y_5 \leqslant u & ① \\ 3y_4 + 5y_5 \leqslant u & ② \\ 6y_5 \leqslant u & ③ \\ y_4 \geqslant 0 & ④ \\ y_5 \geqslant 0 & ⑤ \\ y_4 + y_5 = 1 & \end{array} \right\}$$

## Definition

the normalised polytopes have the following generic form:

$$P = \{x \in \mathbb{R}^m \times \mathbb{R} \mid x \geqslant \mathbf{0}, B^\top x \leqslant \mathbf{1}\}$$
$$Q = \{y \in \mathbb{R}^n \times \mathbb{R} \mid Ay \leqslant \mathbf{1}, y \geqslant \mathbf{0}\}$$

## Example

consider for example $Q$:

$$Q = \left\{ (\frac{y_4}{u}, \frac{y_5}{u}) \mid \begin{array}{ll} 3\frac{y_4}{u} + 3\frac{y_5}{u} \leqslant 1 & ① \\ 3\frac{y_4}{u} + 5\frac{y_5}{u} \leqslant 1 & ② \\ 6\frac{y_5}{u} \leqslant 1 & ③ \\ \vdots & \end{array} \right\}$$

## Observation

- $P$, $Q$ are bounded, hence polytopes
- in this transformation labels are preserved
- every vertex in $P$ ($Q$) has $m$ ($n$) labels as the game is nondegenerated





## Example

points of polytope $P$:

$$\mathbf{0} = (0, 0, 0) \qquad \text{labels } ①, ②, ③$$
$$a = (\frac{1}{3}, 0, 0) \qquad \text{labels } ②, ③, ④$$
$$b = (\frac{2}{7}, \frac{1}{14}, 0) \qquad \text{labels } ③, ④, ⑤$$
$$c = (0, \frac{1}{6}, 0) \qquad \text{labels } ①, ③, ⑤$$
$$d = (0, \frac{1}{8}, \frac{1}{4}) \qquad \text{labels } ①, ④, ⑤$$
$$e = (0, 0, \frac{1}{3}) \qquad \text{labels } ①, ②, ④$$

## Example (cont'd)

points of polytope $Q$:

| | |
|---|---|
| $p = (0, \frac{1}{6})$ | labels ③, ④ |
| $q = (\frac{1}{12}, \frac{1}{6})$ | labels ②, ③ |
| $r = (\frac{1}{6}, \frac{1}{9})$ | labels ①, ② |
| $s = (\frac{1}{3}, 0)$ | labels ①, ⑤ |





# Lemke-Howson (LH) Algorithm

## Algorithm

- INPUT: a nondegenerate bimatrix game
- OUTPUT: one Nash equilibria together with proof of existence

## Notation

- dropping a label $l$ of a vertex $x$ means
  traversing the unique edge that has all the labels of $x$ except $l$
- at the endpoint there is a vertex that has a new label
  this label is picked up

## Method

1. start with the artifical equilibrium $(\mathbf{0}, \mathbf{0})$
2. pick a pure strategy $k \in M \cup N$ that is dropped
3. this label is called the missing label
4. traverse along the unique edge to the endpoint (in $P$ or $Q$)
5. **loop**
   - denote the new vertex pair as $(x, y)$
   - let $l$ denote the label that is picked up
   - if $l = k$, exit loop with Nash equilibrium $(x, y)$
   - otherwise drop $l$ in the other polytope ($Q$ or $P$)

## Corollary

a nondegenerate bimatrix game has an odd number of Nash equilibria

## Proof

endpoints of paths are either Nash equilibria or $(\mathbf{0}, \mathbf{0})$
number of endpoints is even ∎





## Example

some equilibria may remain hidden to the LH algorithm:

$$A = B^\top = \begin{pmatrix} 3 & 3 & 0 \\ 4 & 0 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

## Implementation

- the LH algorithm can be implemented algebraically by pivoting in each step
- pivoting can be handled in a similar way as in the simplex method; this yields a polytime algorithm for each step