Game Theory

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Summary of Last Lecture

Algorithm

Lemke-Howson (LH) Algorithm

- INPUT: a nondegenerate bimatrix game
- OUTPUT: one Nash equilibria together with proof of existence

Notation

- dropping a label l of a vertex x means
  traversing the unique edge that has all the labels of x except l
- at the endpoint there is a vertex that has a new label
  this label is picked up

Method

1. start with the artificial equilibrium $(\mathbf{0}, \mathbf{0})$
2. pick a pure strategy $k \in M \cup N$ that is dropped
3. this label is called the missing label
4. traverse along the unique edge to the endpoint (in P or Q)
5. **loop**
    - denote the new vertex pair as (x, y)
    - let l denote the label that is picked up
    - if $l = k$, exit loop with Nash equilibrium (x, y)
    - otherwise drop l in the other polytope (Q or P)

Implementation

- the LH algorithm can be implemented algebraically by pivoting in each step
- pivoting can be handled in a similar way as in the simplex method; this yields a polytime algorithm for each step

Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games

equilibria of strategic-form games, evolution, resistance, and risk dominance, two-person zero-sum games

efficient computation of Nash equilibria, equilibrium computation for two-player games

complexity of finding Nash equilibria, complexity class PPAD

Nash equilibria and NP-completeness

Definition

a search problem S consists of

1. a set of inputs $I_S \subseteq \Sigma^*$
2. $\forall\ x \in I_S\ \exists$ solution set $S_x \subseteq \Sigma^{|x|^k}$ for some integer k
3. such that $\forall\ x \in I_S\ \forall\ y \in \Sigma^*$ it is decidable in polytime whether $y \in S_x$

a search problem is total if $\forall\ x \in I_S\ S_x \neq \varnothing$

Definition

we write NASH for the problem of finding a Nash equilibrium in a game in strategic form

Example

NASH is a total search problem

NP-completeness of Generalisations

Definition

a bimatrix game Γ represented by payoff matrices A and B is symmetric if $A = B^\top$

Theorem

the following problems are complete for NP (even for symmetric games): given a two-player game Γ in strategic form, does Γ have:

- at least two Nash equilibria?
- a Nash equilibrium in which player i has utility at least a given amount?
- a Nash equilibrium with support of size greater than a given number?
- a Nash equilibrium whose support contains strategy s?
- ...

The Class Polynomial Parity Argument (Directed Case)

Definition — END OF THE LINE

the END OF THE LINE problem is defined as follows:

- GIVEN: two Boolean circuits S and P, each with n input and n output bits, such that

$$P(0^n) = 0^n \qquad S(0^n) \neq 0^n$$

- FIND: an input $\{0,1\}^n$ such that

$$P(S(x)) \neq x \quad \text{or} \quad S(P(x)) \neq x \neq 0^n$$

Definition — PPAD

PPAD (polynomial parity argument, directed case) is the class of total search problems that is reducible to the END OF THE LINE problem

Informal Definition

the class PPAD can be defined as the class of total search problems, where totality follows from an argument like follows

- a directed graph G is defined on a finite but exponentially large set of vertexes
- each vertex has indegree and outdegree at most 1
- given a string x is is easy to check that
  1. $x \in G$
  2. find the adjacent vertexes of x
  3. identify the direction of the edge
- $\exists$ a vertex with no incoming edges that is known (the standard source)
- all vertexes with no outgoing edges, or all sources other than the standard source are solutions

Example

NASH $\in$ PPAD

Succinct Representations of Games

Observation ①

- given an n-player game
- such that each player has the same number of (pure) strategies m
- then representing a game in strategic form needs nm^n numbers

Observation ②

this trivialises any complexity considerations:

- the support enumeration algorithm roughly needs $(2^m)^n$ many steps
- but this is a polynomial algorithm in nm^n

Graphical Games

Definition

a graphical game is a n-person game, with n large, but the utility of each player depends only on the strategies of few other players

- $\exists$ directed graph $G = (\{1, \dots, n\}, E)$
- such that $(i, j) \in E$ implies that the utility of player j depends on the strategy chosen by player i
- $\forall$ (randomised) strategy profiles σ, σ' if $\sigma_j = \sigma'_j$ and $\forall$ $(i, j) \in E$: $\sigma_i = \sigma'_i$, then $u_j(\sigma) = u_j(\sigma')$

Observation ③

given a graphical n-player game Γ such that

- indegree of the graph G at most d
- maximal m pure strategy per player

then Γ needs only nm^{d+1} numbers for its description

Complexity of Nash Equilibria

Definition

SYMMETRIC NASH is the problem of finding a symmetric Nash equilibrium given a symmetric game

Theorem

∃ polynomial reduction from NASH to SYMMETRIC NASH

Proof

on black board ■

Corollary

finding any Nash equilibrium in a symmetric game is as hard as solving NASH

Observation

the existence of a symmetric Nash equilibrium in a symmetric game follows by an adaption of the LH algorithm

NASH is complete for PPAD

Definition

the problem BROUWER, a discrete version of Brouwer's fixpoint theorem: any continuous function f on (let's say) cube has a fixpoint

Theorem

NASH (even for two players) is complete for PPAD

Proof Plan

- BROUWER is complete for PPAD
- reduction from BROUWER to a graphical game Γ with many players
- reduction from Γ to NASH

Final Remark

if P = NP, then also P = PPAD, but P = PPAD need not imply P = NP