Game Theory

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Summary of Last Lecture

Definition

an *n*-person extensive-form game Γ^e is a labelled tree, where also edges are labelled such that

1. each nonterminal node has player label in $\{0, 1, \ldots, n\}$
   nodes labelled with 0 are called chance nodes
   nodes labelled within $\{1, \ldots, n\}$ are called decision nodes
2. edges leaving chance nodes (also called alternatives)
   are labelled with probabilities that sum up to 1
3. player nodes have a second label, the information label
   reflecting the information state
4. each alternative at a player node has a move label
5. each terminal node is labelled with $(u_1, \ldots, u_n)$, the payoff

6. $\forall$ player i, $\forall$ nodes x y z controlled by i $\forall$ alternatives b at x
   - if y and z have the same information state and if y follows x and b
   - $\exists$ node w and some alternative c at w such that z follows w and c
   - w is controlled by player i, w has the same information label as x and c the same move label as b

Recall

the last assertion expresses perfect recall:
whenever a player moves, she remembers all the information she knew earlier

Perfect Information Games

Definition

if no two nodes have the same information state, we say the game has perfect information

Definition (strategy)

- S_i is the set of information states per player i
- D_s is the set of possible moves at $s \in S_i$
- the set of strategies for player i is

$$\bigotimes_{s \in S_i} D_s = \underbrace{D_s \times D_s \times \cdot \times D_s}_{|S_i|\text{-times}}$$

Content

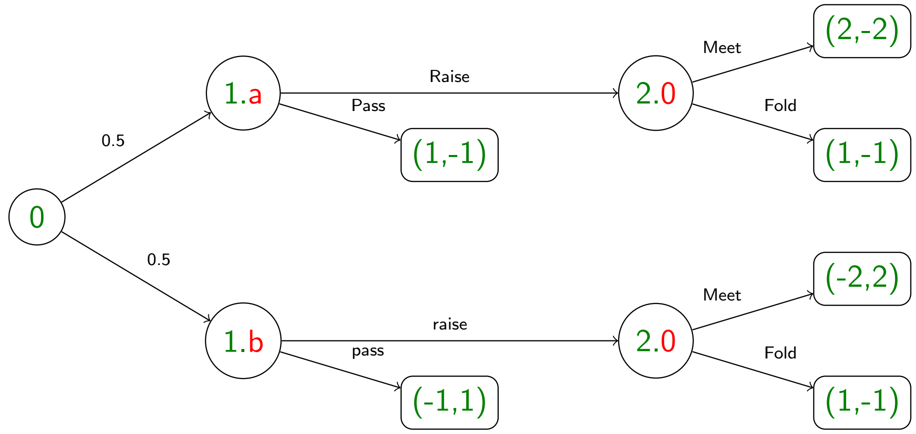
motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games

equilibria of strategic-form games, evolution, resistance, and risk dominance, sequential equilibria of extensive-form games, subgame-perfect equilibria, complexity of finding Nash equilibria, equilibrium computation for two-player games

refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

Example



Strategies

$$\underbrace{\{Rr, Rp, Pr, Pp\}}_{\text{for player 1}} \qquad \underbrace{\{M, F\}}_{\text{for player 2}}$$

Strategic-Form Games

Definition

a strategic-form game is a tuple $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$ such that

1. N is the set of players
2. for each i: C_i is the set of strategies of player i
3. for each i: $u_i \colon \bigotimes_{i \in N} C_i \to \mathbb{R}$ is the expected utility payoff

a strategic-form game is finite if N and each C_i is finite

Example

consider the card game, suppose player 1 plans to use strategy Rp and player 2 plans to use M

$$u_1(Rp, M) = 2 \cdot \frac{1}{2} + -1 \cdot \frac{1}{2} = 0.5$$

$$u_2(Rp, M) = -2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = -0.5$$

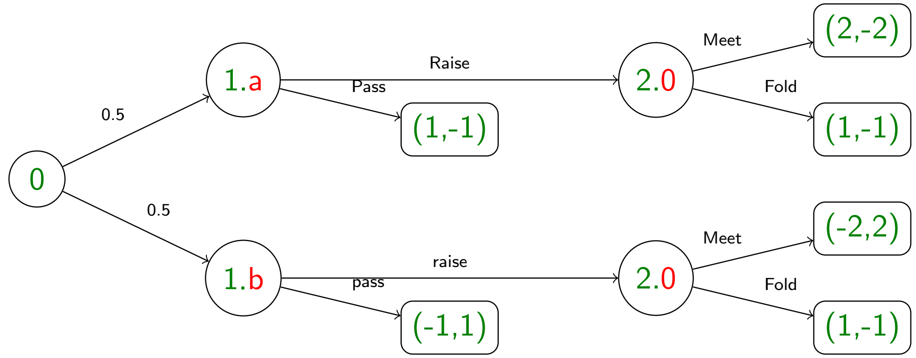
Definition

given a game Γ^e in extensive form, we define the normal representation as strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$:

1. $N = \{1, \ldots, n\}$, if Γ^e is an n-person game
2. for each i: C_i denotes the strategies of each player as defined above
3. we define the expected utility payoff u_i
   - set $C = \bigotimes_{i \in N} C_i$
   - let x be a node in Γ^e
   - let $c \in C$ denote a given strategy profile
   - let $P(x|c)$ denotes the probability that the path of play goes through x, if c is chosen
   - let Ω^* denote the set of all terminal nodes
   - for $x \in \Omega^*$, $w_i(x)$ denotes the payoff for player i
   - set

$$u_i(c) = \sum_{x \in \Omega^*} P(x|c) w_i(x)$$

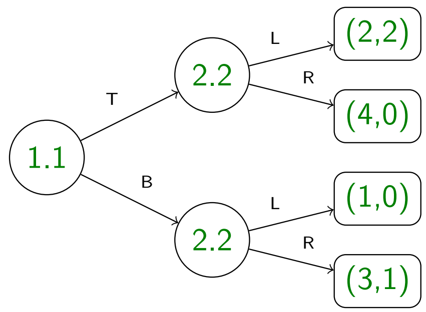
Example

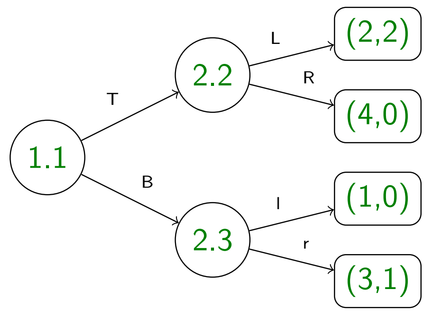


Normal Representation

C_1	C_2: M	F
Rr	$0, 0$	$1, -1$
Rp	$0.5, -0.5$	$0, 0$
Pr	$-0.5, 0.5$	$1, -1$
Pp	$0, 0$	$0, 0$

More Examples





C_1	C_2: L	R
T	$2, 2$	$4, 0$
B	$1, 0$	$3, 1$

C_1	C_2: Ll	Lr	Rl	Rr
T	$2, 2$	$2, 2$	$4, 0$	$4, 0$
B	$1, 0$	$3, 1$	$1, 0$	$3, 1$

Equivalence of Strategic-Form Games

Definition

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are fully equivalent if

- for all players i, exists numbers A_i and B_i
- such that $A_i > 0$
- and $u'_i(c) = A_i u_i(c) + B_i$ for all $c \in C = \bigotimes C_i$

Example

C_1	C_2: x_2	y_2
x_1	$9, 9$	$0, 8$
y_1	$8, 0$	$7, 7$

C_1	C_2: x_2	y_2
x_1	$1, 1$	$0, 0$
y_1	$0, 0$	$7, 7$

not fully equivalent, as (x_1, x_2) is better than (y_1, y_2) in the first game, but not in the second

Definition

- let $C_{-i} = \bigotimes_{j \in N \setminus \{i\}} C_j$
- let (e_{-i}, d_i) denote a strategy profile, such that $e_{-i} \in C_{-i}$ and $d_i \in C_i$
- for any set Z and any $f\colon Z \to \mathbb{R}$, define

$$\mathrm{argmax}_{y \in Z} f(y) = \{y \in Z \mid f(y) = \max_{z \in Z} f(z)\}$$

- let $\eta \in \Delta(C_{-i}) = \{q\colon C_{-i} \to \mathbb{R} \mid \sum_{e_{-i} \in C_{-i}} q(e_{-i}) = 1\}$

Definition

best-response equivalence

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are best-response equivalent if (for all η)

$$\mathrm{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i) = \mathrm{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u'_i(e_{-i}, d_i)$$

Example

the previous defined games are best-response equivalent

(Fully) Reduced Normal Representations

Definition

let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we say d_i and e_i in C_i, are payoff equivalent if

$$u_j(c_{-i}, d_i) = u_j(c_{-i}, e_i) \quad \text{for all } c_{-i} \in C_{-i},\, j \in N$$

Definition

a randomised strategy σ_i is any probability distribution over C_i (denoted $\Delta(C_i)$); i.e., $\sigma(c_i)$ denotes the probability that i choses strategy c_i

Definition

a strategey d_i is randomly redundant if $\exists\, \sigma_i \in \Delta(C_i)$ such that $\sigma_i(d_i) = 0$ and

$$u_j(c_{-i}, d_i) = \sum_{e_i \in C_i} \sigma_i(e_i) u_j(c_{-i}, e_i) \quad \text{for all } c_{-i} \in C_{-i},\, j \in N$$

Definition — strongly dominated

let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we say d_i is strongly dominated for player i, if $\exists$ randomised strategey $\sigma_i \in \Delta(C_i)$ such that

$$u_i(c_{-i}, d_i) < \sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \quad \text{for all } c_{-i} \in C_{-i}$$

Definition — residual game

- let $\Gamma^{(0)} = (N, (C_i)_{i \in N}, (u_i)_{i \in N}) := \Gamma$
- let $\Gamma^{(k)} = (N, (C_i^{(k)})_{i \in N}, (u_i)_{i \in N})$, such that $C_i^{(k)}$ denotes the set of all strategies in $C_i^{(k-1)}$ not strongly dominated in $\Gamma^{(k-1)}$
- clearly $C_i \supseteq C_i^{(1)} \supseteq C_i^{(2)} \supseteq \cdots \supseteq C_i^{(n)} = C_i^{(n+1)}$ as $C_i^{(n)}$ cannot become empty, but is finite
- define $\Gamma^{(\infty)} = \Gamma^{(n)}$
- the strategies $C_i^{(\infty)}$ are called iteratively undominated
- $\Gamma^{(\infty)}$ is the residual game

Example

in the card game, strategy Pp is strongly dominated by $\frac{1}{2}[Rr] + \frac{1}{2}[Rp]$

Example

consider

	C_2		
C_1	x_2	y_2	z_2
a_1	$2,3$	$3,0$	$0,1$
b_1	$0,0$	$1,6$	$4,2$

the resdiual game consists of strategy a_1 and x_2