Game Theory

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Summary of Last Lecture

Definition

the outcome of a game in Pareto efficient if there is no other outcome that would make all players better of

> A game may have equilibria that are inefficient, and a game may have multiple equilibria

Example

prisoner dilemma

C_1	g_2 (C_2)	f_2 (C_2)
g_1	5,5	0,6
f_1	6,0	1,1

- the only equilibrium is $([f_1],[f_2])$ which is inefficient

Example

battle of the sexes

C_1	f_2 (C_2)	s_2 (C_2)
f_1	3,1	0,0
s_1	0,0	1,3

- the game as two pure equilibria
$$([f_1],[f_2]) \qquad ([s_1],[s_2])$$
- and one (inefficient) mixed equilibria
$$(0.75[f_1] + 0.25[s_1], 0.25[f_2] + 0.75[s_2])$$

Computing Nash Equilibria

Theorem

- let $D_i \subseteq C_i$ a nonempty subset of player i strategy set C_i expressing our guess which of i's strategy have positive probability
- if there is an equilibrium with support $\bigotimes_{i \in N} D_i$ then $\exists\ (w_i)_{i \in N}$ such that

$$\sum_{c_{-i} \in C_{-i}} \Big(\prod_{j \in N - \{i\}} \sigma_j(c_j) \Big) u_i(c_{-i}, d_i) = w_i \qquad \forall i \in N, \forall d_i \in D_i \tag{1}$$

$$\sigma_i(e_i) = 0 \qquad \forall i \in N, \forall e_i \in C_i \setminus D_i \tag{2}$$

$$\sum_{d_i \in D_i} \sigma_i(d_i) = 1 \qquad \forall i \in N \tag{3}$$

Theorem

the conditions (1)–(3) imply

$$u_i(\sigma) = \sum_{c_i \in C_i} \sigma(c_i) u_i(\sigma_{-i}, [c_i]) = w_i$$

yielding $\sum_{i \in N}(|C_i| + 1)$ equations in $\sum_{i \in N}(|C_i| + 1)$ unknowns

Theorem

if (1)–(3) plus

$$\sigma_i(d_i) \geqslant 0 \qquad \forall i \in N, \forall d_i \in D_i$$

$$w_i \geqslant \sum_{c_{-i} \in C_{-i}} \Big(\prod_{j \in N - \{i\}} \sigma_j(c_j) \Big) u_i(c_{-i}, e_i) \qquad \forall i \in N, \forall e_i \in C_i \setminus D_i$$

are fulfill, then any solution (σ, w) induces an equilibrium σ such that w_i is the expected pay off to player i

Homework

- Exercise 2.1.
- Exercise 2.2.
- Exercise 3.1.
- Exercise 3.3.
- Exercise 3.4.

Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games

equilibria of strategic-form games, evolution, resistance, and risk dominance, two-person zero-sum games

efficient computation of Nash equilibria

sequential equilibria of extensive-form games, subgame-perfect equilibria, complexity of finding Nash equilibria, equilibrium computation for two-player games

Evolution

Idea

identify good strategies by a biological evolutionary criterion

Definition

- $L_i \subseteq \Delta(C_i)$ of promising randomised strategies
- $\forall$ player i
- $\exists$ i-animals that implement a strategy $\sigma_i \in L_i$
- each i-animal plays the game repeatedly using σ_i
- $\forall$ player $j \neq i$
- let the j-animals randomly choose among the strategies in L_j
- define

$$q_j^k(\sigma_j) = \frac{j\text{-animals that implement } \sigma_j}{\text{all } j\text{-animals}} \qquad \text{(in generation } k)$$

Definition

- define

$$\overline{\sigma}^k_{-i}(c_j) = \sum_{\sigma_j \in L_j} q^k_j(\sigma_j)\sigma_j(c_j) \qquad \forall j \in N, \ \forall c_j \in C_j$$

- set $\overline{\sigma}^k_{-i} = (\overline{\sigma}^k_j)_{j \in N}$
- and $\overline{u}^k_i(\sigma_i) = u_i(\overline{\sigma}^k_{-i}, \sigma_i)$

Definition

the number of children in the next generation $k+1$ depends on the expected payoff:

$$q^{k+1}_i(\sigma_i) = \frac{q^k_i(\sigma_i)\overline{u}^k_i(\sigma_i)}{\sum_{\tau_i \in L_i} q^k_i(\sigma_i)\overline{u}^k_i(\tau_i)}$$

“Definition”

strategies that survive in the end, are good

Resistance

Problem

even strategies that behave poorly can be crucial to determine which strategy reproduces best

Definition

- $\forall$ games Γ is strategic form
- $\forall$ σ, τ equilibria in $\bigotimes_{i \in N} \Delta(C_i)$

the resistance of σ against τ is the largest $\lambda \in [0,1]$ such that $\forall j \in N$:

$$u_i((\lambda\tau_j + (1-\lambda)\sigma_j)_{j \in N - \{i\}}, \sigma_i) \geqslant u_i((\lambda\tau_j + (1-\lambda)\sigma_j)_{j \in N - \{i\}}, \tau_i)$$

Idea

the resistance measure the “evolutionary” strength of one equlibrium against another

Risk Dominance

Definition

an equilibrium σ risk dominates another equlibrium τ if the resistance of σ against τ is greater than the resistance of τ against σ

Example

	C_2	
C_1	x_2	y_2
x_1	9,9	0,8
y_1	8,0	7,7

- resistance of (y_1, y_2) against (x_1, x_2) is $\frac{7}{8}$
- resistance of (x_1, x_2) against (y_1, y_2) is $\frac{1}{8}$

Two-Person Zero-Sum Games

Example

	C_2	
C_1	M	F
Rr	$0, 0$	$1, -1$
Rp	$0.5, -0.5$	$0, 0$
Pr	$-0.5, 0.5$	$1, -1$
Pp	$0, 0$	$0, 0$

Observation

$$u_1(c_1, c_2) = -u_2(c_1, c_2) \qquad \forall c_1 \in \{Rr, Rp, Pr, Pp\} \quad \forall c_2 \in \{M, F\}$$

Definition

a two-person zero-sum game Γ in strategic form is a game
$\Gamma = (\{1,2\}, C_1, C_2, u_1, u_2)$: $u_1(c_1, c_2) = -u_2(c_1, c_2)$ $\forall c_1 \in C_1$, $\forall c_2 \in C_2$

Min-Max Theorem

Theorem

(σ_1, σ_2) is an equlibrium of a finite two-person zero-sum game $\Gamma = (\{1,2\}, C_1, C_2, u_1, -u_1)$ if and only if

$$\sigma_1 \in \text{argmax}_{\tau_1 \in \Delta(C_1)} \min_{\tau_2 \in \Delta(C_2)} u_1(\tau_1, \tau_2)$$

$$\sigma_2 \in \text{argmin}_{\tau_2 \in \Delta(C_2)} \max_{\tau_1 \in \Delta(C_1)} u_1(\tau_1, \tau_2)$$

furthermore if (σ_1, σ_2) an equilibrium of Γ, then

$$u_1(\sigma_1, \sigma_2) = \max_{\tau_1 \in \Delta(C_1)} \min_{\tau_2 \in \Delta(C_2)} u(\tau_1, \tau_2) = \min_{\tau_2 \in \Delta(C_2)} \max_{\tau_1 \in \Delta(C_1)} u(\tau_1, \tau_2)$$

Proof

on blackboard ■

Example

	C_2	
C_1	M	F
Rr	$0, 0$	$1, -1$
Rp	$0.5, -0.5$	$0, 0$
Pr	$-0.5, 0.5$	$1, -1$
Pp	$0, 0$	$0, 0$

- allow only the pure straegies
- we obtain:

$$\max_{c_1 \in \{Rr,Rp,Pr,Pp\}} \min_{c_2 \in \{M,F\}} u_1(c_1, c_2) = \max\{0, 0, -0.5, 0\} = 0$$

$$\min_{c_2 \in \{M,F\}} \max_{c_1 \in \{Rr,Rp,Pr,Pp\}} u_1(c_1, c_2) = \min\{0.5, 1\} = 0.5 \neq 0$$

- recall that Γ doesn't admit a pure equlibrium!

Observe

- the proof of the theorem uses the existence of a Nash equilibrium, this is essential
- in particular we need this for

$$\max_{\tau_1 \in \Delta(C_1)} \min_{\tau_2 \in \Delta(C_2)} u(\tau_1, \tau_2) = \min_{\tau_2 \in \Delta(C_2)} \max_{\tau_1 \in \Delta(C_1)} u(\tau_1, \tau_2)$$

Definition

an optimisation problem is defined as

$$\text{minimise}_{x \in \mathbb{R}^n} f(x) \qquad \text{subject to } g_i(x) \geqslant 0 \quad \forall i \in \{1, \dots, m\}$$

where f, g_1, ..., g_m are functions from $\mathbb{R}^n \to \mathbb{R}$

Observation

two-person zero-sum games and optimisation problems are closely linked

Lemma

the optimisation problem

$$\text{minimise}_{x \in \mathbb{R}^n} f(x) \qquad \text{subject to } g_i(x) \geqslant 0 \quad \forall i \in \{1, \dots, m\}$$

is equivalent to

$$\text{minimise}_{x \in \mathbb{R}^n} (\max_{y \in \mathbb{R}^m_+} f(x) - \sum_{i=1}^{m} y_i g_i(x)) \tag{4}$$

here $R^m_+ = \{(y_1, \dots, y_m) \mid y_i \geqslant 0\}$

Proof

observe that $\max_{y \in \mathbb{R}^m_+}(f(x) - \sum_{i=1}^m y_i g_i(x)) = f(x)$ if the constraints are met, otherwise it is $+\infty$

Definition

the dual of (4) is defined as

$$\text{maximise}_{y \in \mathbb{R}^m_+} (\min_{x \in \mathbb{R}^n} f(x) - \sum_{i=1}^{m} y_i g_i(x))$$

Linear Programming

Definition

a linear programming problem is an optimisation problem such that

$$f(x) = \sum_{j=1}^{n} c_j x_j$$

$$g_i(x) = \sum_{j=1}^{n} a_{ij} x_j - b_i \qquad \forall 1 \leqslant i \leqslant m$$

the optimisation problem (4) becomes

$$\text{minimise}_{x \in \mathbb{R}^n} \sum_{j=1}^{n} c_j x_j \quad \text{subject to} \sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \quad \forall i \in \{1, \dots, m\} \tag{5}$$

and its dual becomes

$$\text{maximise}_{y \in \mathbb{R}^m_+} \sum_{i=1}^{m} y_i b_i \quad \text{subject to} \sum_{i=1}^{n} y_i a_{ij} = c_i \quad \forall i \in \{1, \dots, m\} \tag{6}$$

Duality Theorem of Linear Programming

Theorem

suppose $\exists\, x = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} a_{ij} x \geqslant b_i \qquad \forall 1 \leqslant i \leqslant m$$

and $\exists$ at least one nonnegative vector $y = (y_1, \dots, y_m) \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{m} y a_{ij} = c_k \qquad \forall 1 \leqslant j \leqslant n$$

then

- the minimisation problem (5) and the maximisation problem (6) both have optimal solutions
- at these optimal solutions the values of objective functions

  $$\sum_{j=1}^{n} c_j x_j \quad \text{and} \quad \sum_{i=1}^{m} y_i b_i$$

  are equal