1.

a) We claim to define $\mathcal{M} \models \varphi$ for sentences, but in the fourth step, we use the definition with respect to a formula with free variable, namely $\varphi(x)$.

b) The fourth case has to be defined as follows:

$$\mathcal{M} \models \exists x\varphi \qquad \text{if } \mathcal{M}_C \models \varphi(c) \text{ for some } c \in \mathcal{V}(\mathcal{M})$$

where the definition of $\mathcal{M}_C$ and $\mathcal{V}(\mathcal{M})$ can be found on page 61 in the book.

c) If suffices to add one case: $\mathcal{M} \models s = t$ if and only if $s^{\mathcal{M}}$ and $t^{\mathcal{M}}$ denote the same element in the universe of $\mathcal{M}$.

2.

a)
- $\varphi_1^S :\Leftrightarrow \forall x\forall z\forall u(\neg Q(x, f(x), z) \vee P(g(x, z, u), x, f(x), u))$
- $\varphi_2^S :\Leftrightarrow \forall y\forall z((\neg R(a, z) \vee \neg R(a, y) \vee R(a, f(y, z))) \wedge (\neg R(a, z) \vee \neg R(a, y) \vee R(y, f(y, z))) \wedge (\neg R(a, z) \vee \neg R(a, y) \vee R(z, f(y, z))))$
- $\varphi_3 :\Leftrightarrow \forall x\forall y(S(y, f(x, y)) \wedge S(f(x, y), g(x, y)) \wedge S(x, h(x, y)) \wedge S(h(x, y), i(x, y)))$

where $a$, $f$, $g$, $h$, $i$ are newly introduced Skolem constants or Skolem functions, respectively.

b) See Section 3.3.2 in the book on page 116.

c) Suppose $\mathcal{V} = \{c\}$ for some constant symbol $c$. Then the Herbrand vocabulary if $\mathcal{V}$ itself and as Herbrand universe $U$ we get the set $\{c\}$. To finish the definition of the Herbrand structure $\mathcal{M}$, we need to set the valuation of all relation symbols in $\mathcal{V}$. As there are none, we are done.

3.

a) Let $C = \{c_1, c_2, c_3, \dots\}$ be a set of fresh constants. We define $\mathsf{P}$ as follows:
- set $T_0 = \Gamma$
- enumerate the set of all $\mathcal{V}^+$-sentences: $\varphi_1, \varphi_2, \varphi_3, \dots$
- define $T_{m+1}$ based on $T_m$ and consider sentence $\varphi_{m+1}$; assume $T_m$ has only used finitely many constants from $C$
- if $T_m \cup \{\neg\varphi_{m+1}\}$ is consistent, set:

$$T_{m+1} = T_m \cup \{\neg\varphi_{m+1}\}$$

- if $T_m \cup \{\neg\varphi_{m+1}\}$ is not consistent, then $T_m \cup \{\varphi_{m+1}\}$ is consistent
- in this case suppose $\varphi_{m+1} \neq \exists x\psi(x)$, then:

$$T_{m+1} = T_m \cup \{\varphi_{m+1}\}$$

- otherwise $T_{m+1} = T_m \cup \{\varphi_{m+1}\} \cup \{\psi(c_i)\}$ for fresh $c_i \in C$

b) Let $\mathcal{V}+ = \mathcal{V} \cup C$. We define the structure $\mathcal{M}^+$ as a $\mathcal{V}^+$-structure such that:

- the universe of $\mathcal{M}^+$ is a set $U^+$ of closed $\mathcal{V}^+$-terms
- in $U^+$ we identify all terms $s$, $t$ such that $T^+ \vdash s = t$
- set $c^{\mathcal{M}^+} = t \in U^+$, whenever $T^+ \vdash t = c$
- set $f^{\mathcal{M}^+}(t_1, \ldots, t_n) = s$, whenever $T^+ \vdash f(t_1, \ldots, t_n) = s$
- set $(t_1, \ldots, t_n) \in R^{\mathcal{M}^+}$, if $T^+ \vdash R(t_1, \ldots, t_n)$

4. The explicit definitions for $\mathsf{a}$ and $\mathsf{sub}$ are given on page 304 in the book. Here we present the definition of the factorial function:

$$0! = 1 \qquad (n+1)! = h(n, n!) ,$$

where $h(x, y) = \mathsf{p}^2_1(x, y) \times \mathsf{p}^2_2(x, y)$ and $x \times y$ is defined as follows: $0 \times y = i(y) = 0$ and $(x + 1) \times y = j(x, y, x \times y) = \mathsf{a}(y, x \times y)$ with $i(y) = \mathsf{Z}(\mathsf{p}^1_1(y))$, $j(x, y, z) = \mathsf{a}(\mathsf{p}^3_2(x, y, z), \mathsf{p}^3_3(x, y, z))$.

5.

| **statement** | **yes** | **no** |
|---|---|---|
| Given an embedding $f \colon \mathcal{M} \to \mathcal{N}$, let $a$ be an element in $\mathcal{M}$ and let $\varphi(x)$ be a quantifier-free formula, containing $=$ with one free variable $x$. Then $\mathcal{N} \models \varphi(f(a))$ doesn't imply $\mathcal{M} \models \varphi(a)$. | ☐ | ☑ |
| Every embedding is injective. | ☑ | ☐ |
| For any vocabulary $\mathcal{V}$ and any $\mathcal{V}$-structure $\mathcal{M}$, $\mathsf{Th}(\mathcal{M})$ is complete and consistent. | ☑ | ☐ |
| For finite $\mathcal{M}$ we have that $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M} \equiv \mathcal{N}$. | ☑ | ☐ |
| For any sentence $\varphi$ we have: $\varphi$ is satisfiable if and only if $\varphi$ has a Herbrand model. | ☐ | ☑ |
| Let $\Gamma$ be a countable set of formulas, if $\Gamma$ is consistent, then $\Gamma$ has a countable model. | ☑ | ☐ |
| The set of all 17-ary recursive functions is not recursive enumerable. | ☐ | ☑ |
| For each $n$ there exists arithmetic sets $A$ that are $\Sigma_n$ but not $\Pi_n$ and vice versa. | ☑ | ☐ |
| We have $\mathsf{LOGSPACE} \subseteq \mathsf{NLOGSPACE} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{NPSPACE}$. | ☑ | ☐ |
| For any existential second-order sentence $\varphi$ and any $\mathcal{V}$-structure $\mathcal{M}$ such that $\mathcal{V}$ is finite, the problem $\mathcal{M} \models \varphi$ is decidable by a nondeterministic TM in polynomial time. | ☑ | ☐ |