1. Consider propositional logic and the following formula:

$$\neg[((Q \vee R) \wedge (P \to Q)) \to ((R \to \neg Q) \vee (P \wedge R))]$$

   a) Use resolution to decide whether the formula is satisfiable or not. If the formula is unsatisfiable provide a complete refutation. If it is satisfiable provide a satisfying assignment. (3 pts)

   b) Consider a finite (undirected) graph $G = (V, E)$, where the set of vertexes is defined as $V = \{a, b, c, d, e\}$ and the set of edges defined as

   $$E = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}\}$$

   If possible formalise the following expression in propositional logic: *Node e is reachable from node a with a path of at most length 3 that passed through node c.* If it is impossible, explain why. (3 pts)

2. Consider the following sentences:

   ① *Each dragon is happy if all its children can fly.*

   ② *Dragons can fly if and only if they are green or red.*

   ③ *A dragon is green if it is the child of at least one green dragon.*

   ④ *Red dragons spit fire.*

   ⑤ *There are green dragons which cannot spit fire.*

   a) For each of the assertions above, give a sentence in first-order logic that formalises it. Use the following constants, functions and predicates:
      - constants: $\mathsf{green}$, $\mathsf{red}$
      - functions: $\mathsf{color}^1$
      - predicates: $\mathsf{dragon}^1$, $\mathsf{happy}^1$, $\mathsf{fly}^1$, $\mathsf{child}^2$, $\mathsf{spitfire}^1$, $\mathsf{E}^2$

      The atomic formula $\mathsf{child}(x, y)$ is to be interpreted as "$x$ is a child of $y$" and the formula $\mathsf{E}(x, y)$ as "$x$ is equal to $y$". (5 pts)

   b) Combine the formulas described in item a) to a description of the world as above and show that your formalisation is satisfiable by giving a model of it. (3 pts)

3. Consider the following (incorrect) definitions of substructure and elementary substructure. (The definition makes use of introduced notation.)
   - Let $\mathcal{M}$, $\mathcal{N}$ be structures. Assume that the vocabulary of $\mathcal{M}$ is a subset of the vocabulary of $\mathcal{N}$, that the universe of $\mathcal{M}$ and $\mathcal{N}$ is the same and that $\mathcal{M}$ interprets its vocabulary in the same way as $\mathcal{N}$. Then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.
   - $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ if there exists an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

a) Give the correct definitions. (3 pts)

b) Explain by example why the above definitions are incorrect: Give two structures $\mathcal{M}$, $\mathcal{N}$ which are (elementary) substructures with respect to the faulty definitions, but not for the correct one. (4 pts)

c) Show the following assertion: Given two isomorphic structures $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent. (4 pts)

4. Consider the following sentences in prenex normal form:

$$\begin{aligned}
\varphi_1 &:\Leftrightarrow \forall x \exists y \forall z \forall u \exists w (Q(x,y,z) \rightarrow P(w,x,y,u)) \\
\varphi_2 &:\Leftrightarrow \exists x \forall y \forall z \exists w (R(x,z) \wedge R(x,y) \rightarrow (R(x,w) \wedge R(y,w) \wedge R(z,w))) \\
\varphi_3 &:\Leftrightarrow \forall x \forall y \exists z \exists u \exists v (S(y,z) \wedge (S(z,u) \wedge (S(x,v) \wedge S(v,u))))
\end{aligned}$$

a) Define the Skolemisations $\varphi_i^S$ of the sentences $\varphi_i$, $i = 1,2,3$ given above. (6 pts)

b) Give the crucial idea of the following claim: *For any SNF formula $\varphi$ (containing $=$) there exists a formula $\varphi'$ (without $=$) such that $\varphi$ is satisfiable iff $\varphi'$ is satisfiable.* (4 pts)

c) Give an example of a finite Herbrand structure. (2 pts)

5. Consider the following function $\mathsf{fun}\colon \mathbb{N}^2 \rightarrow \mathbb{N}$:

$$\mathsf{fun}(0,n) = n{+}1 \qquad \mathsf{fun}(m{+}1,0) = \mathsf{fun}(m,1) \qquad \mathsf{fun}(m{+}1,n{+}1) = \mathsf{fun}(m,\mathsf{fun}(m{+}1,n))$$

Decide whether this function is primitive recursive or not. An informal argument suffices. (3 pts)

6. Determine whether the statements on the answer sheet are true or false. **Every correct answer is worth 1 points and every wrong -1 points**. (10 pts)

- Literal embeddings preserve universal sentences.
- Any consistent theory is complete.
- First-order logic is complete, i.e., for any sentence $\varphi$, if $\models \varphi$, then $\vdash \varphi$.
- Let $\Gamma$ be a countable set of sentences. Then $\Gamma \models \varphi$ if and only if there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.
- For graphs, reachability is expressible as existential, first-order formula.
- Let $A$ be a definable subset of $\mathbb{N}$. If $A$ is definable by an existential formula, then $A$ is recursive.
- For any existential second-order sentence $\varphi$ and any $\mathcal{V}$-structure $\mathcal{M}$ such that $\mathcal{V}$ is finite, then the problem $\mathcal{M} \models \varphi$ is in $\mathsf{P}$.
- Consider the complexity classes $\mathsf{NP}$ and $\mathsf{co-NP} = \{L \mid \sim L \in \mathsf{NP}\}$. In the arithmetical hierarchy we have $\Sigma_1 = \mathsf{NP}$ and $\Pi_1 = \mathsf{co-NP}$.
- Second order logic is complete, but not compact.
- The game `Soduku` is as difficult as the satisfiability problem `SAT`, both are complete for $\mathsf{PSPACE}$.