1. a) The formula is satisfiable. The only assignment ν that satisfies the formula is given as $\nu(P) = 0$ and $\nu(Q) = \nu(R) = 1$.

   b) For each edge, we employ a propositional variable X_{ij}, where $i, j \in \{a, b, c, d, e\}$. Then the node e is reachable from node a by a path of length at most 3 that goes through node c iff the following formula P is satisfiable:

$$(X_{ac} \wedge X_{ae}) \vee \bigvee_{i \in \{a,b,c,d,e\}} (X_{ai} \wedge X_{ic} \wedge X_{ce}) \vee \bigvee_{i \in \{a,b,c,d,e\}} (X_{ac} \wedge X_{ci} \wedge X_{ie}) \; .$$

   It remains to link P with the actual description of the graph. For that we make sure that all edge variables X_{ij} are evaluated to 1 iff there exists an edge from i to j in the graph G. Making use of truth constants $\top$ and $\bot$ (that can be defined if not present) this is easy. We call this description D. Then the formula $D \to P$ is satisfiable iff a node e is reachable from node a with a path of at most length 3 that passed through node c in the graph G.

2. a) The five statements can be formalised as follows:

   ① $\forall x[(\mathsf{dragon}(x) \wedge \forall y(\mathsf{child}(y,x) \to \mathsf{fly}(y))) \to \mathsf{happy}(x)]$
   ② $\forall x\ [\mathsf{dragon}(x) \to (\mathsf{fly}(x) \leftrightarrow (\mathsf{E}(\mathsf{color}(x), \mathsf{green}) \vee \mathsf{E}(\mathsf{color}(x), \mathsf{red})))]$
   ③ $\forall x \exists y\ [(\mathsf{dragon}(x) \wedge \mathsf{dragon}(y) \wedge \mathsf{child}(x,y) \wedge \mathsf{E}(\mathsf{color}(y), \mathsf{green})) \to \mathsf{E}(\mathsf{color}(x), \mathsf{green})]$
   ④ $\forall x\ [(\mathsf{dragon}(x) \wedge \mathsf{E}(\mathsf{color}(x), \mathsf{red})) \to \mathsf{spitfire}(x)]$
   ⑤ $\exists x\ [\mathsf{dragon}(x) \wedge \mathsf{E}(\mathsf{color}(x), \mathsf{green}) \wedge \neg\mathsf{spitfire}(x)]$

   b) By taking the conjunction of the formulas ①, . . . ,⑤ we get the single formula

$$① \wedge ② \wedge ③ \wedge ④ \wedge ⑤$$

   which formalises the rules of the world described above.
   To show that the formalisation is satisfiable we define a interpretation $\mathcal{M}$ as follows:
   - $U_{\mathcal{M}} = \{\text{Mushu}, \text{GREEN}, \text{RED}\}$
   - $\mathsf{green}^{\mathcal{M}} = \text{GREEN}$
     $\mathsf{red}^{\mathcal{M}} = \text{RED}$
   - $\mathsf{color}^{\mathcal{M}}(\text{Mushu}) = \text{GREEN}$
     $\mathsf{color}^{\mathcal{M}}(\text{GREEN}) = \text{GREEN}$
     $\mathsf{color}^{\mathcal{M}}(\text{RED}) = \text{RED}$
   - $\mathsf{dragon}^{\mathcal{M}} = \{\text{Mushu}\}$
     $\mathsf{fly}^{\mathcal{M}} = \mathsf{happy}^{\mathcal{M}} = \{\text{Mushu}\}$
     $\mathsf{E}^{\mathcal{M}} = \{(\text{GREEN}, \text{GREEN})\}$
     $\mathsf{child}^{\mathcal{M}} = \mathsf{spitfire}^{\mathcal{M}} = \varnothing$

   It can be easily checked that the above interpretation is indeed a model of the formula ① $\wedge \ldots \wedge$ ⑤.

3. a) See Definition 2.6.2 in the book.

b) Consider two structures $\mathcal{M}$ and $\mathcal{N}$ based on signatures $\mathcal{V}_{\mathcal{M}} = \{P\}$ and $\mathcal{V}_{\mathcal{N}} = \{P, Q\}$ respectively such that

$$a, b \in P^{\mathcal{M}},\ a, b \in P^{\mathcal{N}}(b), \text{ and } a \in Q^{\mathcal{N}}$$

Then $\mathcal{M}$ is substructure of $\mathcal{N}$ with respect to the (faulty) definition, but for the correct definition the vocabularies $\mathcal{V}_{\mathcal{M}}$ and $\mathcal{V}_{\mathcal{N}}$ would have to coincide and Q ought to be interpreted in $\mathcal{M}$.

c) Elementary equivalence means that the same formulas are modelled in both structures. If two structures are isomorph they are equal upto renaming of domain elements. Hence obviously the same formulas are true.

4. a) We obtain the following Skolem normal forms:

$$\begin{aligned}
&\varphi_1^S :\Leftrightarrow \forall x \forall z \forall u(\neg Q(x, f(x), z) \vee P(g(x, z, u), x, f(x), u)) \\
&\varphi_2^S :\Leftrightarrow \forall y \forall z((\neg R(a, z) \vee \neg R(a, y) \vee R(a, f(y, z))) \wedge \\
&\qquad (\neg R(a, z) \vee \neg R(a, y) \vee R(y, f(y, z))) \wedge (\neg R(a, z) \vee \neg R(a, y) \vee R(z, f(y, z)))) \\
&\varphi_3 :\Leftrightarrow \forall x \forall y(S(y, f(x, y)) \wedge S(f(x, y), g(x, y)) \wedge S(x, h(x, y)) \wedge S(h(x, y), g(x, y)))
\end{aligned}$$

where a, f, g, h are newly introduced Skolem constants or Skolem functions, respectively.

b) See Section 3.3.2 in the book on page 116.

c) Suppose $\mathcal{V} = \{c, P\}$ for some constant symbol c and some predicate symbol P. Then the Herbrand vocabulary if $\mathcal{V}$ itself and as Herbrand universe U we get the set $\{c\}$. To finish the definition of the Herbrand structure $\mathcal{M}$, we need to set the valuation of all relation symbols in $\mathcal{V}$. Let us set $c \in P^{\mathcal{M}}$, i.e. $P(c)$ is true in $\mathcal{M}$. Then we are done.

5. This is the Ackermann function, which is well-known not be primitive recursive.

6.

statement	**yes**	**no**
Literal embeddings preserve universal sentences.		✓
Any consistent theory is complete.		✓
First-order logic is complete, i.e., for any sentence φ, if $\models \varphi$, then $\vdash \varphi$.	✓	
Let Γ be a countable set of sentences. Then $\Gamma \models \varphi$ if and only if there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.	✓	
For graphs, reachability is expressible as existential, first-order formula.		✓
Let A be a definable subset of $\mathbb{N}$. If A is definable by an existential formula, then A is recursive.		✓
For any existential second-order sentence φ and any $\mathcal{V}$-structure $\mathcal{M}$ such that $\mathcal{V}$ is finite, then the problem $\mathcal{M} \models \varphi$ is in P.		✓
Consider the complexity classes NP and $\mathsf{co} - \mathsf{NP} = \{L \mid \sim L \in \mathsf{NP}\}$. In the arithmetical hiearchy we have $\Sigma_1 = \mathsf{NP}$ and $\Pi_1 = \mathsf{co} - \mathsf{NP}$.		✓
Second order logic is complete, but not compact.		✓
The game `Soduku` is as difficult as the satisfiability problem `SAT`, both are complete for PSPACE.		✓