Logic (master program)

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Summary of Last Lecture

> Model theory is a technique, that given
> a model of a system and a formal property
> checks whether this property holds
> for that model

Definition — definable

let $\mathcal{M}$ be a structure with universe U

- $\varphi(\mathcal{M}) = \{(a_1, \dots, a_n) \in U^n \mid \mathcal{M} \models \varphi(a_1, \dots, a_n)\}$
- $\varphi(\mathcal{M})$ is called $\mathcal{V}$-definable subset of $\mathcal{M}$

Definition

let A, B be (possible infinite) sets, we set $|A| \leqslant |B|$
if $\exists\ f : A \to B$, such that f is injective

Definition

- we say A and B have the same size (denoted $|A| = |B|$)
  if $|A| \leqslant |B|$ and $|B| \leqslant |A|$
- we write $|A| < |B|$ if $|A| \leqslant |B|$ but not $|A| = |B|$

Theorem — Bernstein's Theorem

sets A and B have the same size if and only if there exists a bijection between A and B

Definition

a set A

- is denumerable if $\exists$ bijection $f : A \to \mathbb{N}$
- is countable if A is finite or denumerable, otherwise it is uncountable

Theorem

- if the vocabulary $\mathcal{V}$ is countable, so is the set of all $\mathcal{V}$-formulas
- the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable

Corollary

there are uncountable many properties (over a countable vocabulary) that cannot be defined

Content

introduction, propositional logic, semantics, formal proofs, resolution (propositional)

first-order logic, semantics, structures , theories and models, formal proofs, Herbrand theory, resolution (first-order), completeness of first-order logic, properties of first-order logic

introduction to computability, introduction to complexity, finite model theory

beyond first order: modal logics in a general setting, higher-order logics, introduction to Isabelle

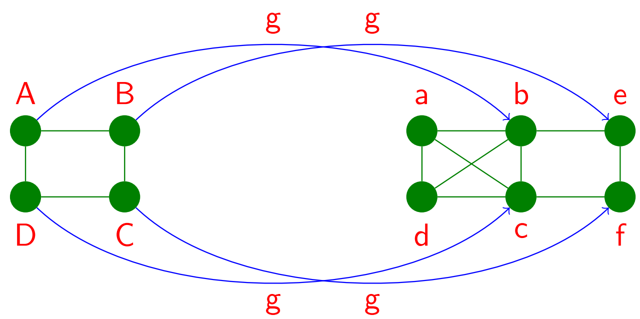
Definition (embedding)

- let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-structures, a function $f\colon \mathcal{M} \to \mathcal{N}$ preserves the $\mathcal{V}$-formula $\varphi(x_1, \dots, x_n)$ if for each tuple $a_1, \dots, a_n$ in $\mathcal{M}$

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \quad \text{implies} \quad \mathcal{N} \models \varphi(f(a_1), \dots, f(a_n))$$

- if f preserves all $\mathcal{V}$-formulas that are literals, then f is a (literal) embedding
- if f preserves all $\mathcal{V}$-formulas, then f is an elementary embedding

Example



g is a literal embedding

Definition (existential)

- a quantifier-free formula does not contain the quantifiers $\forall$, $\exists$
- an existential formula is of form

$$\exists y_1 \dots \exists y_m \varphi(\overline{x}, y_1, y_2, \dots, y_m)$$

  such that $\varphi(\overline{x}, y_1, y_2, \dots, y_m)$ is quantifier free

Lemma

let $f\colon \mathcal{M} \to \mathcal{N}$ be an embedding; then for any quantifier-free formula $\varphi(\overline{x})$ and any tuple $\overline{a}$ of elements in $\mathcal{M}$

$$\mathcal{M} \models \varphi(\overline{a}) \quad \text{if and only if} \quad \mathcal{N} \models \varphi(f(\overline{a}))$$

Proof

by induction on φ ∎

Lemma

embeddings preserve existential formulas

Proof

- let $f\colon \mathcal{M} \to \mathcal{N}$ be an embedding; let $\varphi(\overline{x})$ be existential
- set $\varphi(\overline{x}) = \exists y_1 \dots \exists y_m \psi(\overline{x}, y_1, y_2, \dots, y_m)$
- by the semantics of $\exists$, $\mathcal{M} \models \varphi(\overline{a})$ means

$$\mathcal{M} \models \psi(\overline{a}, \overline{b}) \quad \text{for some } \overline{b}$$

- from previous lemma

$$\mathcal{N} \models \psi(f(\overline{a}), f(\overline{b}))$$

- hence

$$\mathcal{N} \models \varphi(f(\overline{a}))$$

∎

Fact

any embedding $f\colon \mathcal{M} \to \mathcal{N}$ is injective, as $\mathcal{M} \models a \neq b$ if and only if $\mathcal{N} \models f(a) \neq f(b)$

Isomorphism

Lemma

if $f\colon \mathcal{M} \to \mathcal{N}$ is onto, then f is a literal embedding if and only if f is an elementary embedding

Proof Sketch

on proves by induction that f and f^{-1} preserve each formula ■

Definition — isomorphism

- a bijective $f\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism, if it preserves every formula
- if an isomorphism exists, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic ($\mathcal{M} \cong \mathcal{N}$)

Definition — elementarily equivalent

if $\mathcal{M}$ and $\mathcal{N}$ model the same sentences, then $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent ($\mathcal{M} \equiv \mathcal{N}$)

Fact

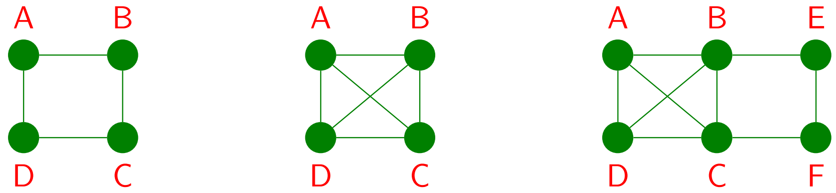
if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$

Definition — substructure

$\mathcal{M}$ is substructure of $\mathcal{N}$ ($\mathcal{M} \subseteq \mathcal{N}$) if

1. $\mathcal{M}$ and $\mathcal{N}$ are structures that have the same vocabulary
2. the universe of $\mathcal{M}$ is a subset of the universe of $\mathcal{N}$
3. $\mathcal{M}$ interprets the vocabulary in the same way as $\mathcal{N}$

Example



subgraph of G — induced subgraph of G — G

Definition

if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{N}$ is an extension of $\mathcal{M}$

Definition

- a formula $\varphi(\overline{x})$ is preserved under extensions when

$$\text{if} \quad \mathcal{M} \subseteq \mathcal{N} \quad \text{then} \quad \mathcal{M} \models \varphi(\overline{a}) \quad \text{implies} \quad \mathcal{N} \models \varphi(\overline{a})$$

- a formula $\varphi(\overline{x})$ is preserved under substructures when

$$\text{if} \quad \mathcal{M} \subseteq \mathcal{N} \quad \text{then} \quad \mathcal{N} \models \varphi(\overline{a}) \quad \text{implies} \quad \mathcal{M} \models \varphi(\overline{a})$$

Theorem

- quantifier-free formulas are preserved under substructures and extensions
- existential formulas are preserved under extensions

Definition

a universal formula is of form $\forall y_1 \dots \forall y_m \varphi(\overline{x}, y_1, y_2, \dots, y_m)$ such that $\varphi(\overline{x}, y_1, y_2, \dots, y_m)$ is quantifier free

Elementary Substructure

Theorem

universal formulas are preserved under substructures

Definition — elementary substructure

$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ ($\mathcal{M} \prec \mathcal{N}$) if id: $\mathcal{M} \to \mathcal{N}$ is an elementary embedding

Example

- let $\mathcal{N} = (\mathbb{N}, \mathsf{succ})$ denote the structure of the naturals, where succ denotes the successor
- let $\mathcal{M}$ be the substructure of $\mathcal{N}$ on universe $\{1, 2, 3, 4, \dots\}$
- define $f\colon \mathcal{M} \to \mathcal{N}$ as $f(x) = x + 1$
- f is an isomorphism, hence $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}$
- but id is not an elementary embedding, let $\varphi(x) :\Leftrightarrow \neg\exists y (s(y) = x)$

$$\mathcal{M} \models \varphi(1) \qquad \mathcal{N} \not\models \varphi(1)$$

Diagrams

Definition (diagram)

- the elementary diagram ($\mathrm{ED}(\mathcal{M})$) of a structure $\mathcal{M}$ is the set of all $\mathcal{V}(\mathcal{M})$-sentences that hold in $\mathcal{M}_C$
- the (literal) diagram ($\mathrm{D}(\mathcal{M})$) of a structure $\mathcal{M}$ is the set of all literals in $\mathrm{ED}(\mathcal{M})$

Theorem

the following assertions are equivalent

1. $\mathcal{M}$ can be (literal) embedded into $\mathcal{N}$
2. $\exists$ expansion $\mathcal{N}'$ of $\mathcal{N}$ such that $\mathcal{N}' \models \mathrm{D}(\mathcal{M})$
3. $\exists$ extension $\mathcal{M}'$ of $\mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{N}$

Theorem

let $\mathcal{V}$ be finite, for any finite $\mathcal{M}$ $\exists$ formula $\varphi_{\mathcal{M}}$ such that for any finite $\mathcal{V}$-structure $\mathcal{N}$, $\mathcal{N} \models \varphi_{\mathcal{M}}$ if and only if $\mathcal{M} \cong \mathcal{N}$

Proof

- let $\overline{a} = \{a_1, a_2, \ldots, a_n\}$ be the universe of $\mathcal{M}$
- let $\varphi(\overline{a})$ denote the conjunction of all literals in $\mathrm{D}(\mathcal{M})$
- let $\exists\overline{x}\varphi(\overline{x}) :\Leftrightarrow \exists x_1 \exists x_2 \ldots \exists x_n \varphi(\overline{x})$
- let ψ_n denote

$$\forall x_1 \forall x_2 \ldots \forall x_{n+1} \Big(\bigvee_{i \neq j} (x_i = x_j)\Big)$$

- we set $\varphi_{\mathcal{M}} :\Leftrightarrow \psi_n \wedge \exists\overline{x}\varphi(\overline{x})$
- suppose $\mathcal{N} \models \varphi_{\mathcal{M}}$,
- by the previous theorem, $\mathcal{M}$ is embeddable by an f into $\mathcal{N}$, moreover $|\mathcal{N}| \leqslant n$
- hence f is bijective, and thus f is an bijective, elementary embedding, or an isomorphism ∎

Theory of a Structure

Corollary

for finite $\mathcal{M}$ we have that $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M} \equiv \mathcal{N}$

Fact

the corollary is false for infinite structures, which makes model theory for infinite structures interesting (and boring for finite ones)

Definition (theory of $\mathcal{M}$)

the theory of a $\mathcal{V}$-structure $\mathcal{M}$ is defined as

$$\mathrm{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi \text{ and } \varphi \text{ is a } \mathcal{V}\text{-sentence}\}$$

Definition (model)

for any set Γ of $\mathcal{V}$-sentences, a model of Γ is a $\mathcal{V}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$; the class of all models is denoted as $\mathrm{Mod}(\Gamma)$

Definition (complete theory)

Γ is a complete $\mathcal{V}$-theory if for any $\mathcal{V}$-sentence φ, either φ or $\neg\varphi$ is in Γ and Γ does not contain φ and $\neg\varphi$

Theorem

for any $\mathcal{V}$-structure $\mathcal{M}$, $\mathrm{Th}(\mathcal{M})$ is a complete theory

Definition (consistent)

a set of sentences Γ is called consistent if no contradictions can be derived from it

Definition (theory)

- a theory is any consistent set of sentences
- if T is a theory, then $\mathrm{Mod}(T)$ is called elementary class

Observation

- let T denote a theory, and call T maximal, if any proper superset is not consistent
- then a maximal theory is complete and vice versa

> Model theory studies theories and models and the interaction between them. Understanding the theory of a structure lends insight into the structure. On the other hand, understanding the models of a theory lends insight into the theory