Logic (master program)

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Summary of Last Lecture

Definition (embedding)

- let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-structures, a function $f\colon \mathcal{M} \to \mathcal{N}$ preserves the $\mathcal{V}$-formula $\varphi(x_1, \ldots, x_n)$ if for each tuple $a_1, \ldots, a_n$ in $\mathcal{M}$

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \quad \text{implies} \quad \mathcal{N} \models \varphi(f(a_1), \ldots, f(a_n))$$

- if f preserves all $\mathcal{V}$-formulas that are literals, then f is a (literal) embedding
- if f preserves all $\mathcal{V}$-formulas, then f is an elementary embedding

Lemma
embeddings preserve existential formulas

Lemma
if $f\colon \mathcal{M} \to \mathcal{N}$ is onto, then f is a literal embedding if and only if f is an elementary embedding

Isomorphism

Definition (isomorphism)

- a bijective $f\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism, if it preserves every formula
- if an isomorphism exists, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic ($\mathcal{M} \cong \mathcal{N}$)

Definition (substructure)
$\mathcal{M}$ is substructure of $\mathcal{N}$ ($\mathcal{M} \subseteq \mathcal{N}$) if

1. $\mathcal{M}$ and $\mathcal{N}$ are structures that have the same vocabulary
2. the universe of $\mathcal{M}$ is a subset of the universe of $\mathcal{N}$
3. $\mathcal{M}$ interprets the vocabulary in the same way as $\mathcal{N}$

Lemma

- existential formulas are preserved under extensions
- universal formulas are preserved under substructures

Elementary Equivalence

Definition (elementarily equivalent)
if $\mathcal{M}$ and $\mathcal{N}$ model the same sentence, then $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent ($\mathcal{M} \equiv \mathcal{N}$)

Theorem
let $\mathcal{V}$ be finite, for any finite $\mathcal{M}$ $\exists$ formula $\varphi_{\mathcal{M}}$ such that for any finite $\mathcal{V}$-structure $\mathcal{N}$, $\mathcal{N} \models \varphi_{\mathcal{M}}$ if and only if $\mathcal{M} \cong \mathcal{N}$

Corollary
for finite $\mathcal{M}$ we have that $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M} \equiv \mathcal{N}$

Theories and Models

Definition

theory of $\mathcal{M}$

the theory of a $\mathcal{V}$-structure $\mathcal{M}$ is defined as

$$\mathsf{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi \text{ and } \varphi \text{ is a } \mathcal{V}\text{-sentence}\}$$

Definition

complete theory

Γ is a complete $\mathcal{V}$-theory if for any $\mathcal{V}$-sentence φ, either φ or $\neg\varphi$ is in Γ and Γ does not contain φ and $\neg\varphi$

Theorem

for any $\mathcal{V}$-structure $\mathcal{M}$, $\mathsf{Th}(\mathcal{M})$ is a complete theory

- Exercise 2.4.
- Exercise 2.9.
- Exercise 2.20.
- Exercise 2.22.
- Exercise 2.26.

Content

introduction, propositional logic, semantics, formal proofs, resolution (propositional)

first-order logic, semantics, structures , theories and models, formal proofs, Herbrand theory, resolution (first-order), completeness of first-order logic, properties of first-order logic

introduction to computability, introduction to complexity, finite model theory

beyond first order: modal logics in a general setting, higher-order logics, introduction to Isabelle

Rules for Derivations

premise	conclusion	name
$\varphi \in \Gamma$	$\Gamma \vdash \varphi$	assumption
$\Gamma \vdash \varphi \land \Gamma \subset \Gamma'$	$\Gamma' \vdash \varphi$	monotonicity
$\Gamma \vdash \varphi$	$\Gamma \vdash \neg\neg\varphi$	double negation
$\Gamma \vdash \varphi, \Gamma \vdash \psi$	$\Gamma \vdash (\varphi \land \psi)$	$\land$-introduction
$\Gamma \vdash (\varphi \land \psi)$	$\Gamma \vdash \varphi$	$\land$-elimination
$\Gamma \vdash (\varphi \land \psi)$	$\Gamma \vdash (\psi \land \varphi)$	$\land$-symmetry
$\Gamma \vdash \varphi$	$\Gamma \vdash \varphi \lor \psi$	$\lor$-introduction
$\Gamma \vdash (\varphi \lor \psi), \Gamma \cup \{\varphi\} \vdash \theta, \Gamma \cup \{\psi\} \vdash \theta$	$\Gamma \vdash \theta$	$\lor$-elimination
$\Gamma \vdash (\varphi \lor \psi)$	$\Gamma \vdash (\psi \lor \varphi)$	$\lor$-symmetry
$\Gamma \cup \{\varphi\} \vdash \psi$	$\Gamma \vdash (\varphi \to \psi)$	$\to$-introduction
$\Gamma \vdash (\varphi \to \psi), \Gamma \vdash \varphi$	$\Gamma \vdash \psi$	$\to$-elimination

More Rules

premise	conclusion	name
$\Gamma \vdash \varphi$	$\Gamma \vdash (\varphi)$	()-introduction
$\Gamma \vdash (\varphi)$	$\Gamma \vdash \varphi$	()-elimination
$\Gamma \vdash ((\varphi \wedge \psi) \wedge \theta)$	$\Gamma \vdash (\varphi \wedge \psi \wedge \theta)$	$\wedge$-parentheses
$\Gamma \vdash ((\varphi \vee \psi) \vee \theta)$	$\Gamma \vdash (\varphi \vee \psi \vee \theta)$	$\vee$-parentheses
$\Gamma \vdash (\varphi \vee \psi)$	$\Gamma \vdash \neg(\neg\varphi \wedge \neg\psi)$	$\vee$-definition
$\Gamma \vdash \neg(\neg\varphi \wedge \neg\psi)$	$\Gamma \vdash (\varphi \vee \psi)$	
$\Gamma \vdash (\varphi \to \psi)$	$\Gamma \vdash (\neg\varphi \vee \psi)$	$\to$-definition
$\Gamma \vdash (\neg\varphi \vee \psi)$	$\Gamma \vdash (\varphi \to \psi)$	
$\Gamma \vdash (\varphi \leftrightarrow \psi)$	$\Gamma \vdash (\varphi \to \psi) \wedge (\psi \to \varphi)$	$\leftrightarrow$-definition
$\Gamma \vdash (\varphi \to \psi) \wedge (\psi \to \varphi)$	$\Gamma \vdash (\varphi \leftrightarrow \psi)$	

Yet More Rules

premise	conclusion	name
$\Gamma \vdash \varphi(t)$	$\Gamma \vdash \exists x \varphi(x)$	$\exists$-introduction
$\Gamma \vdash \varphi(c)$	$\Gamma \vdash \forall x \varphi(x)$	$\forall$-introduction
$\Gamma \vdash \varphi \to \psi$	$\Gamma \vdash \exists x \varphi(x) \to \exists x \psi(x)$	$\exists$-distribution
$\Gamma \vdash \varphi \to \psi$	$\Gamma \vdash \forall x \varphi(x) \to \forall x \psi(x)$	$\forall$-distribution
$\Gamma \vdash Q_1 x (Q_2 y \varphi)$	$\Gamma \vdash Q_1 x Q_2 y \varphi$	parentheses rule
	$\Gamma \vdash t = t$	reflexivity
$\Gamma \vdash \varphi(t), \Gamma \vdash t = t'$	$\Gamma \vdash \varphi(t')$	equality substitution

- t is a term
- c is constant $\notin \Gamma$
- $Q \in \{\forall, \exists\}$

extend by definition rules

Formal Proof

Definition — formal proof

- a formal proof is a finite sequence of statements $\Gamma \vdash \varphi$
- each statement follows from previous ones, by the stated rules
- we say φ s derived from Γ if there is a formal proof of $\Gamma \vdash \varphi$

Theorem — soundness

if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Corollary

if both $\underbrace{\{\varphi\} \vdash \psi \text{ and } \{\psi\} \vdash \varphi}_{\text{provably equivalent}}$ then $\varphi \equiv \psi$

Proof (of Theorem)

by induction on the number of steps, for each rule one shows that the inference is sound

Case — $\exists$-distribution

suppose $\mathcal{M} \models \varphi \to \psi$ and $\mathcal{M} \models \exists x \varphi$, we show $\mathcal{M} \models \exists x \psi$

1. x is not a free variable of φ

   hence $\varphi \equiv \exists x \varphi$, so from $\mathcal{M} \models \exists x \varphi$, we conclude $\mathcal{M} \models \varphi$ and from $\mathcal{M} \models \varphi \to \psi(x)$, we conclude $\mathcal{M} \models \psi(x)$; but then $\mathcal{M}_C \models \psi(a)$ for $a \in \mathcal{V}(\mathcal{M})$, hence $\mathcal{M} \models \exists x \psi(x)$

2. x is free variable of φ, but not of ψ

   assumption $\mathcal{M} \models \varphi(x) \to \psi$ asserts that $\mathcal{M} \models \varphi(a) \to \psi$ for all $a \in \mathcal{V}(\mathcal{M})$, since $\mathcal{M} \models \exists x \varphi(x)$, we obtain $\mathcal{M} \models \psi$ and thus $\mathcal{M} \models \exists x \psi$

3. x is free in φ and ψ

   from $\mathcal{M} \models \varphi(x) \to \psi(x)$, we obtain $\mathcal{M} \models \varphi(a) \to \psi(a)$ for all $a \in \mathcal{V}(\mathcal{M})$, as $\mathcal{M} \models \exists x \varphi(x)$, we have $\mathcal{M} \models \varphi(c)$ for some $c \in \mathcal{V}(\mathcal{M})$, hence $\mathcal{M} \models \psi(c)$ and conclusively $\mathcal{M} \models \exists x \psi(x)$

■

Theorem — closure theorem

let $\varphi(x_1, \ldots, x_n)$ be a formula and let $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$ its universal closure, then $\Gamma \vdash \varphi(x_1, \ldots, x_n)$ if and only if $\Gamma \vdash \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where Γ is a set of sentences

Proof

- suppose $\Gamma \vdash \varphi(x)$ such that only the variable x occurs free
- then $\Gamma \vdash \varphi(c)$ for some fresh constant c
- hence $\Gamma \vdash \forall x \varphi(x)$ by $\forall$-introduction ■

Theorem

let x, y be variables that do not occur in $\varphi(z)$

- $\forall x \varphi(x)$ and $\forall y \varphi(y)$ are provably equivalent
- $\exists x \varphi(x)$ and $\exists y \varphi(y)$ are provably equivalent

Definition — prenex normal form

- a formula φ is in prenex normal form if it has the form

$$Q_1 x_1 \ldots Q_n x_n \psi \qquad Q_i \in \{\forall, \exists\}$$

  ψ is quantifier-free
- if ψ is a conjunction of disjunctions of literals, we say φ is in conjunctive (prenex) normal form

Theorem

any first-order formula is transformable into conjunctive normal form