University of Innsbruck **Institute of Computer Science**
Test 2A **January 15, 2010**

Functional Programming **WS 2009/2010** **LVA 703018**

Solutions

This test consists of three exercises. *Explain your answers*. The available points for each item are written in the margin.

[8] **1.** Reduce the λ-term

$$(\lambda p.p\ (\lambda x\ y.x))\ ((\lambda x\ y\ f.f\ x\ y)\ (\lambda x.x)\ \mathsf{Y})$$

to normal form using the leftmost outermost reduction strategy. Here, Y denotes the λ-term $\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$.

Solution.

$$\begin{aligned}
&(\lambda p.p\ (\lambda x\ y.x))\ ((\lambda x\ y\ f.f\ x\ y)\ (\lambda x.x)\ \mathsf{Y})\\
&= \underline{(\lambda p.p\ (\lambda x\ y.x))\ ((\lambda x\ y\ f.f\ x\ y)\ (\lambda x.x)\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))))}\\
&\to_\beta \underline{(\lambda x\ y\ f.f\ x\ y)\ (\lambda x.x)}\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))\ (\lambda x\ y.x)\\
&\to_\beta \underline{(\lambda y\ f.f\ (\lambda x.x)\ y)\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))}\ (\lambda x\ y.x)\\
&\to_\beta \underline{(\lambda f.f\ (\lambda x.x)\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))))\ (\lambda x\ y.x)}\\
&\to_\beta \underline{(\lambda x\ y.x)\ (\lambda x.x)}\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))\\
&\to_\beta \underline{(\lambda y\ x.x)\ (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))}\\
&\to_\beta \lambda x.x
\end{aligned}$$

[8] **2.** Consider the OCaml functions

```
let rec (@) xs ys = match xs with []     -> ys
                                | x::xs -> x::(xs@ys)

let rec rev = function []     -> []
                     | x::xs -> rev xs @ [x]
```

Prove by induction that

$$\mathtt{rev}(\mathtt{rev}\ xs) = xs$$

for all lists xs. You may use the equation

$$\mathtt{rev}(xs\ \mathtt{@}\ ys) = (\mathtt{rev}\ ys)\ \mathtt{@}\ (\mathtt{rev}\ xs) \qquad (\star)$$

Solution. We us induction over xs to show the property

$$P(xs) = (\mathtt{rev}(\mathtt{rev}\ xs) = xs)$$

Base Case ($xs = \mathtt{[]}$). $P(\mathtt{[]})$ is shown by the derivation:

$$\mathtt{rev}(\mathtt{rev}\ \mathtt{[]}) = \mathtt{[]}$$

Step Case ($xs = z :: zs$). The IH is $P(zs) = (\texttt{rev}(\texttt{rev}\ zs) = zs)$. $P(z :: zs)$ is shown by the derivation:

$$\begin{aligned}
\texttt{rev}(\texttt{rev}(z :: zs)) &= \texttt{rev}(\texttt{rev}\ zs\ \texttt{@}\ \texttt{[}z\texttt{]}) \\
&= \texttt{rev}\ \texttt{[}z\texttt{]}\ \texttt{@}\ \texttt{rev}(\texttt{rev}\ zs) && \text{by } (\star) \\
&= \texttt{rev}\ \texttt{[}z\texttt{]}\ \texttt{@}\ zs && \text{by IH} \\
&= (\texttt{rev}\ \texttt{[]}\ \texttt{@}\ \texttt{[}z\texttt{]})\ \texttt{@}\ zs \\
&= (\texttt{[]}\ \texttt{@}\ \texttt{[}z\texttt{]})\ \texttt{@}\ zs \\
&= \texttt{[}z\texttt{]}\ \texttt{@}\ zs \\
&= z :: (\texttt{[]}\ \texttt{@}\ zs) \\
&= z :: zs
\end{aligned}$$

[5] **3.** (a) Consider the type inference problem $E \triangleright (\lambda x.\mathsf{add}\ x\ 1)\ y : \alpha_0$ using the typing environment $E = \{\mathsf{add} : \mathsf{int} \to \mathsf{int} \to \mathsf{int}, 1 : \mathsf{int}, y : \mathsf{int}\}$. Apply the type inference rules of $\mathcal{I}$, to compute a resulting unification problem. Give all the intermediate computation steps.

Solution.

$$E \rhd (\lambda x.\mathsf{add}\ x\ 1)\ y : \alpha_0$$
$$\Rightarrow_{(\mathsf{app})}$$
$$E \rhd \lambda x.\mathsf{add}\ x\ 1 : \alpha_1 \to \alpha_0; E \rhd y : \alpha_1$$
$$\Rightarrow_{(\mathsf{con})}$$
$$E \rhd \lambda x.\mathsf{add}\ x\ 1 : \alpha_1 \to \alpha_0;$$
$$\mathsf{int} \approx \alpha_1$$
$$\Rightarrow_{(\mathsf{abs})}$$
$$E' = E, x : \alpha_2 \rhd \mathsf{add}\ x\ 1 : \alpha_3;$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3$$
$$\Rightarrow_{(\mathsf{app})}$$
$$E' \rhd \mathsf{add}\ x : \alpha_4 \to \alpha_3; E' \rhd 1 : \alpha_4;$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3$$
$$\Rightarrow_{(\mathsf{con})}$$
$$E' \rhd \mathsf{add}\ x : \alpha_4 \to \alpha_3;$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4$$
$$\Rightarrow_{(\mathsf{app})}$$
$$E' \rhd \mathsf{add} : \alpha_5 \to \alpha_4 \to \alpha_3; E' \rhd x : \alpha_5;$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4$$
$$\Rightarrow_{(\mathsf{con})}$$
$$E' \rhd x : \alpha_5;$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3$$
$$\Rightarrow_{(\mathsf{con})}$$
$$\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3; \alpha_2 \approx \alpha_5$$

[4] (b) Solve the following unification problem.

$$\mathsf{int} \approx \alpha_1$$
$$\alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3$$
$$\mathsf{int} \approx \alpha_4$$
$$\mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3$$
$$\alpha_2 \approx \alpha_5$$

Solution.

$$\begin{gathered}
\mathsf{int} \approx \alpha_1; \alpha_1 \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3; \alpha_2 \approx \alpha_5 \\
\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_1/\mathsf{int}\}} \\
\mathsf{int} \to \alpha_0 \approx \alpha_2 \to \alpha_3; \mathsf{int} \approx \alpha_4; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3; \alpha_2 \approx \alpha_5 \\
\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_2/\alpha_5\}} \\
\mathsf{int} \to \alpha_0 \approx \alpha_5 \to \alpha_3; \mathsf{int} \approx \alpha_4; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_3 \\
\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_4/\mathsf{int}\}} \\
\mathsf{int} \to \alpha_0 \approx \alpha_5 \to \alpha_3; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \mathsf{int} \to \alpha_3 \\
\Rightarrow^{(\mathsf{d}_2)+}_{\iota} \\
\mathsf{int} \approx \alpha_5; \alpha_0 \approx \alpha_3; \mathsf{int} \approx \alpha_5; \mathsf{int} \approx \mathsf{int}; \mathsf{int} \approx \alpha_3 \\
\Rightarrow^{(\mathsf{t})}_{\iota} \\
\mathsf{int} \approx \alpha_5; \alpha_0 \approx \alpha_3; \mathsf{int} \approx \alpha_5; \mathsf{int} \approx \alpha_3 \\
\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_0/\alpha_3\}} \\
\mathsf{int} \approx \alpha_5; \mathsf{int} \approx \alpha_5; \mathsf{int} \approx \alpha_3 \\
\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_5/\mathsf{int}\}} \\
\mathsf{int} \approx \mathsf{int}; \mathsf{int} \approx \alpha_3 \\
\Rightarrow^{(\mathsf{t})}_{\iota} \\
\mathsf{int} \approx \alpha_3 \\
\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_3/\mathsf{int}\}} \\
\square
\end{gathered}$$

Combining the intermediate substitutions yields the resulting substitution

$$\{\alpha_0/\mathsf{int}, \alpha_1/\mathsf{int}, \alpha_2/\mathsf{int}, \alpha_3/\mathsf{int}, \alpha_4/\mathsf{int}, \alpha_5/\mathsf{int}\}$$