Game Theory

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Institute of Computer Science @ UIBK

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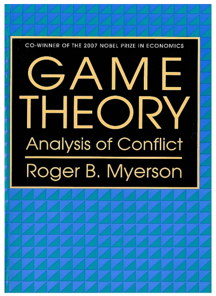
Organisation

Time and Place

Tuesday, 8:15-10:00, HS 10

week 1	October 6	week 8	November 24
week 2	October 13	week 9	December 1
week 3	October 20	week 10	December 15
week 4	October 27	week 11	January 12
week 5	November 3	week 12	January 19
week 6	November 10	week 13	January 26
week 7	November 17	first exam	February 2

Literature & Online Material



Literature

Roger B. Myerson
Game Theory: Analysis of Conflict
Harvard University Press, 1991
ISBN: 0-674-34116-3

Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay V. Vazirani (ed.)
Algorithmic Game Theory
Cambridge University Press, 2007
ISBN 978-0-521-87282-9

Online Material

Transparencies and homework will be available from IP starting with 138.232 after the lecture; exercises and solutions will be discussed during the lecture

Homework & Exam

- officially there are no exercises as this course is labelled VO
- however, without homework I have to talk all the time, this is too exhausting
- the homework assignments will be discussed in the lecture; participation can only positively influence the final grade

Office Hours

Wednesday, 13:00–15:00, 3N01, IfI Building

Outline

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games, Nash equilibrium

two-person zero-sum games, Bayesian equilibria, sequential equilibra of extensive-form games, computing Nash equilibria, subgame-perfect equilibra

efficient computation of Nash equilibria, complexity class PPAD, complexity of Nash equilibria, refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

(perhaps) guest lecture: introduction to mechanism design and auctions

What is Game Theory?

game theory is conceivable as the study of mathematical models of conflict and cooperation between intelligent and rational decision-makers

Why is this part of a computer science major?

- why not, it is part of computer science major at RWTH Aachen
- why not, Ariel Rubinstein writes

  > *There are many similarities between logic and game theory. Whereas logic is the study of truth and inference, game theory is the study of strategic considerations.*

  and Scott Shenker writes

  > *the Internet is the equilibrium, we just have to identify the game*

History of Game Theory

Zermelo	Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels	1913	✓
Borel	La Théorie du Jeu et les Équations Intègrales à Noyau Symètrique	1921	✓
von Neumann	Zur Theorie der Gesellschaftsspiele	1928	✓
von Neumann, Morgenstern	Theory of Games and Economic Behaviour	1944	✓
Nash	The Bargaining Problem	1950	
Harsanyi	A Simplified Bargaining Model for the n-Person Cooperative Game	1963	
Selten	Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit	1965	

Logic and Games

Games in the History of Logic

it can be argued that already Aristotle made the connection between (propositional) logic and games; syllogism are introduced and argued for in the context of a specific game, a debate

Logical Games

- these are two-player games: ∀belard and ∃loise
- the domain of the game is set Ω
- ∀belard and ∃loise play by choosing elements from Ω
- an infinite sequence of elements of Ω is called a play
- a finite sequence is called position
- for each position a, τ(a) decides the next player
- $W_{\forall belard}$ ($W_{\exists loise}$) denotes the set of plays where ∀belard (∃loise) wins

Example — Hintikka game

consider the language of first-order, the game is played on formulas

- legal moves for $\forall$belard:
  - given $A \wedge B$, $\forall$belard chooses either A or B
  - given $\forall x F(x)$, $\forall$belard chooses some instance a
- legal moves for $\exists$loise:
  - given $A \vee B$, $\exists$loise chooses either A or B
  - given $\exists x F(x)$, $\exists$loise chooses some instance a
- the game for $\neg F$ is the dual of F
- $\exists$loise wins if and only if the encountered atomic formula is true

Lemma

given a formulas F, call the Hintikka game $\mathcal{G}(F)$; then $\exists$loise has a winning strategy if and only if F is valid (in first-order logic)

Remark

idea can be extended to temporal logics
most successful for LTL, CTL, CTL*, μ-calculus ...

Content

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Decision-Theoretic Foundations

Definition — game

- a game refers to any social situation involving two or more individuals
- the players are supposed to be rational and intelligent
- rational means the player makes decisions consistently in pursuit of her own objective
- intelligent means the player can make the same inferences about the game that we can make

Definition — probability distribution

let Z be a finite set, the probability distributions $\Delta(Z)$ over Z are defined as follows:

$$\Delta(Z) = \{q \colon Z \to \mathbb{R} \mid \sum_{y \in Z} q(y) = 1 \text{ and } \forall z \in Z \ q(z) \geqslant 0\}$$

Definition — lottery

- let Ω denote set of possible states, and let X denote the set of prizes
- Ω and X are finite
- a lottery is a function f that maps Ω to $\Delta(X)$ such that

$$\sum_{x \in X} f(x|t) = 1 \qquad t \in \Omega$$

- the set of lotteries is defined as follows:

$$L = \{f \mid f \colon \Omega \to \Delta(X)\}$$

- let t be a state, $f(\cdot|t)$ denotes the probability distribution over X in t:

$$f(\cdot|t) = (f(x|t))_{x \in X} \in \Delta(X)$$

Example — roulette lotteries

suppose $\Omega = \varnothing$, then lotteries depend only on objective unknowns that can be assigned probabilities, for example the prize is determined by a coin toss

Example — horse lotteries

suppose the only probabilities that occur in $\Delta(X)$ are 1 or 0, then the final prize (the pay-out if our horse wins) depends only on the state (the strength of the horse), such states are called subjective unknowns

Definition — event

- an event is a (non-empty) subset of Ω
- the sets of all events Ξ is defined as

$$\Xi = \{S \mid S \subseteq \Omega \text{ and } S \neq \varnothing\}$$

Definition

let f, g be lotteries and S an event

- we write $f \succcurlyeq_S g$ if f is at least as desirable as g (with respect to the states in S)
- $f \sim_S g$ iff $f \succcurlyeq_S g$ and $g \succcurlyeq_S f$
- $f \succ_S g$ iff $f \succcurlyeq_S g$ and $f \not\preccurlyeq_S g$

Definition

let $\alpha \in [0,1]$, let f, g be lotteries

- the compound lottery $\alpha f + (1-\alpha)g$ is the defined as:

$$\alpha f(x|t) + (1-\alpha)g(x|t)$$

- the lottery $[x]$ always get prize x for sure:

$$[x](y|t) = 1 \quad \text{if } y = x \qquad\qquad [x](y|t) = 0 \quad \text{if } y \neq x$$

  where $t \in \Omega$

Question

what is $\alpha[x] + (1-\alpha)[y]$?

Answer

it denotes the lottery that gives price x with probability α and prize y with probability $1-\alpha$

How-to Formalise Our Preferences

for a given event S, the relation $\succcurlyeq_S$ seems to fulfil:

- totality
  in acting rational, we order our preferences
- relevance
  if two lotteries f, g coincide for all $s \in S$, f and g are equivalent
- monotonicity
  a higher probability of getting a better lottery is better
- continuity
  suppose we prefer f over g, g over h; then we should be able to express g in terms of f and h

Axiomatic Presentation ①

Axiom (totality)

- $f \succcurlyeq_S g$ or $g \succcurlyeq_S f$
- if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $f \succcurlyeq_S h$

Axiom (relevance)

if for all $t \in S$: $f(\cdot|t) = g(\cdot|t)$, then $f \sim_S g$

Axiom (monotonicity)

if $f \succ_S h$ and $0 \leqslant \beta < \alpha \leqslant 1$, then $\alpha f + (1-\alpha)h \succ_S \beta f + (1-\beta)h$

Axiom (continuity)

if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $\exists\, \gamma \in [0,1]$ such that $g \sim_S \gamma f + (1-\gamma)h$

How-to Formalise Our Preferences (cont'd)

further the relation $\succcurlyeq_S$ seems to fulfil

- substitution
  if the decision maker chooses between options and there are mutually exclusive events, such that regardless of the event one option is favoured, then this option has to be favoured before the event is learnt
- substitution comes in two flavours
  either the 'event' is a random variable (an objective unknown) or a state (a subjective unknowns)
- interest
  the decision maker is never indifferent
- state neutrality
  all prizes are valued the same under different events
- the last axiom can be dropped

Axiomatic Presentation ②

Axiom (objective substitution)
if $e \succcurlyeq_S f$ and $g \succcurlyeq_S h$ and $\alpha \in [0,1]$, then $\alpha e + (1-\alpha)g \succcurlyeq_S \alpha f + (1-\alpha)h$

Axiom (strict objective substitution)
if $e \succ_S f$ and $g \succcurlyeq_S h$ and $\alpha \in (0,1]$, then $\alpha e + (1-\alpha)g \succ_S \alpha f + (1-\alpha)h$

Axiom (subjective substitution)
if $f \succcurlyeq_S g$ and $f \succcurlyeq_T g$ and $S \cap T = \varnothing$, then $f \succcurlyeq_{S \cup T} g$

Axiom (strict subjective substitution)
if $f \succ_S g$ and $f \succ_T g$ and $S \cap T = \varnothing$, then $f \succ_{S \cup T} g$

Axiomatic Presentation ③

Axiom (interest)
$\forall\ t \in \Omega,\ \exists\ x, y \in X$ such that $[y] \succ_{\{t\}} [x]$

Axiom (state neutrality)
$\forall\ r, t \in \Omega$, if $f(\cdot|r) = f(\cdot|t)$ and $g(\cdot|r) = g(\cdot|t)$, and $f \succcurlyeq_{\{r\}} g$, then $f \succcurlyeq_{\{t\}} g$

Definition conditional-probability
a conditional-probability function is any function $p\colon \Xi \to \Delta(\Omega)$ such that

$$p(t|S) = 0 \quad \text{if } t \notin S \qquad\qquad \sum_{r \in S} p(r|S) = 1$$

Definition utility function
a utility function is any function from $u\colon X \times \Omega \to \mathbb{R}$

Expected Utility Maximisation Theorem

Definition
let p denote a conditional-probability function and u any utility function, then the expected utility determined by lottery f is defined as:

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in X} u(x,t) f(x|t)$$

Theorem
the axioms (with state neutrality) are satisfied if and only if there exists a (state-independent) utility function u and a conditional-probability function such that

1. $\max_{x \in X} u(x,t) = 1$ and $\min_{x \in X} u(x,t) = 0$
2. $p(R|T) = p(R|S)p(S|T)\ \forall\ R, S, T$ so that $R \subseteq S \subseteq T$ and $S \neq \varnothing$ where $p(R|S) = \sum_{r \in R} p(r|S)$
3. $f \succcurlyeq_S g$ if and only if $E_p(u(f)|S) \geqslant E_p(u(g)|S)$