Game Theory

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Summary of Last Lecture

Example

consider the following game Γ

C_1 \ C_2	L	R
T	$0, 0$	$0, -1$
B	$1, 0$	$-1, 3$

the unique equilibrium is

$$(\frac{3}{4}[T] + \frac{1}{4}[B], \frac{1}{2}[L] + \frac{1}{2}[R])$$

based on Γ we can define a Bayesian game Γ^b (using parameter ϵ with Bayesian equilibrium (σ_1, σ_2)

$$\sigma_1(\cdot|\alpha) = \begin{cases} [T] & \alpha > \frac{2+\epsilon}{8+\epsilon^2} \\ [B] & \alpha < \frac{2+\epsilon}{8+\epsilon^2} \end{cases} \qquad \sigma_2(\cdot|\beta) = \begin{cases} [L] & \beta > \frac{4-\epsilon}{8+\epsilon^2} \\ [B] & \beta < \frac{4-\epsilon}{8+\epsilon^2} \end{cases}$$

Observation

if $\epsilon \longrightarrow 0$, the Bayesian equilibrium (σ_1, σ_2) becomes the unique equilibrium in the game with complete information

Auctions

not really a new idea

- used by the Babylonians (500 BC)
- first Roman fire brigade offered to buy the burning house and only extinguished the fire if the offer was accepted
- after having killed Emperor Pertinax, the Prätorian Guard auctioned off the Roman Empire (193)
- Johann Wolfgang von Goethe sold a manuscript through a second-price auction (1797)
- biggest revenue yet was generated by the US FCC spectrum auctions (1994–2008)

First Price Auctions

- let F be an increasing and differentiable function
- let M be the maximal value of the object
- and let β denote the bidding function
  assumed to be increasing and differentiable

Lemma

assume types/bids are uniformly distributed $(F(y) = \frac{y}{M})$:

$$\beta(v_i) = (1 - \frac{1}{n})v_i \qquad \forall v_i \in [0, M]$$

Definition

- an auction where the private values are independent is called independent private values
- if the value of the object is the same for all bidders, but the bidders have different private information, the auction is an common value auction

Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, common knowledge, Bayesian games, incomplete information, Nash equilibrium

two-person zero-sum games, Bayesian equilibria, sequential equilibria of extensive-form games, subgame-perfect equilibria

(efficient) computation of Nash equilibria, complexity class PPAD, complexity of Nash equilibria, refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

Mixed and Behavioural Strategies

given a game Γ^e in extensive form, we defined the normal representation of Γ^e as a strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$

Definition

1. $\forall\ i \in N$, S_i denotes information states of i; set $S^* = \bigcup_{i \in N} S_i$
2. $\forall\ s \in S_i$, let Y_s denote the set of nodes that belong to i in information state s
3. D_s is the set of possible moves at s
4. define strategies $C_i = \prod_{s \in S_i} D_s$

Definition

a mixed-strategy profile of Γ^e is any randomised strategy of the normal representation of Γ^e; set of profiles is

$$\prod_{i \in N} \Delta(C_i)$$

Multiagent Representation

let Γ^e be a game in extensive form

Definition

multiagent representation

the multiagent representation of Γ^e is the strategic-form game

$$\Gamma' = (S^*, (D_s)_{s \in S^*}, (v_s)_{s \in s^*})$$

1. S^* is the collection of all information states
2. D_s is the set of moves at each state s
3. $v_s \colon \prod_{s \in S^*} D_s \to \mathbb{R}$, if for any $(c_i)_{i \in N}$ $\forall\ i \in N$, $t \in S_i$: $c_i(t) = d_t$, then $v_r((d_s)_{s \in S^*} = u_i((c_i)_{i \in N})$

Definition

a behavioural-strategy profile of Γ^e is any randomised strategy of the multiagent representation of Γ^e; set of profiles is

$$\prod_{s \in S^*} \Delta(D_s) = \prod_{i \in N} \prod_{s \in S_i} \Delta(D_s)$$

Definition

- $\forall$ player i
- $\forall$ pure strategies $c_i \in C_i$
- $\forall$ information state $s \in S_i$

s and c_i are compatible if $\exists\ c_{-i} \in C_{-i}$ such that $\sum_{x \in Y_s} P(x|c) > 0$

Definition

$C_i^*, C_i^\dagger$

1. $C_i^*(s)$ denotes all strategies in C_i compatible with information state s
2. $C_i^\dagger(d_s, s) = \{c_i \in C_i^*(s) \mid c_i(s) = d_s\}$

Definition

a behavioural strategy $\sigma_i = (\sigma_{i,s})_{s \in S_i}$ is behavioural representation of a mixed strategy $\tau_i \in \Delta(C_i)$ if

$$\sigma_{i,s}(d_s)\Big(\sum_{c_i \in C_i^*(s)} \tau(c_i)\Big) = \sum_{e_i \in C_i^\dagger(d_{i,s})} \tau_i(e_i)$$

Definition

1. two mixed strategies are behaviourally equivalent if they share a common behavioural representation
2. two mixed strategy profiles τ, τ' (in $\prod_{i \in N} \Delta(C_i)$) are behaviourally equivalent if $\forall\ i$ τ_i and τ_i' are behaviourally equivalent

Theorem

if Γ^e is a game with perfect recall, then any two mixed strategies in $\Delta(C_i)$ that are behaviourally equivalent are payoff equivalent

Remark

the condition on perfect recall is crucial

Definition

for any behavioural strategy $\sigma_i = (\sigma_{i,s})_{s \in S_i}$, the mixed representation $\tau_i \in \Delta(C_i)$ satisfies

$$\tau_i(c_i) = \prod_{s \in S_i} \sigma_{i,s}(c_i(s)) \qquad \forall\ c_i$$

Equilibria in Behavioural Strategies

Definition

a (Nash) equilibrium of an extensive-form game is any equilibrium σ of the multiagent representation such that the mixed representation of σ is an equilibrium of the normal representation

Theorem

if Γ^e admits perfect recall and τ is an equilibrium of the normal representation of Γ^e, then any behavioural representation of τ is an equilibrium (of the multiagent representation of Γ^e

Theorem

$\forall\ \Gamma^e$ with perfect recall $\exists$ a Nash equilibrium (of Γ^e)

Games with Perfect Information

Recall

a game with perfect information is any extensive-form game in which each information state is unique

Definition

- a node $x \in \Gamma^e$ is a subroot if for every $s \in S^*$ either $Y_s \cap \Gamma^e|_x = \varnothing$ or $Y_s \subseteq \Gamma^e|_x$
- a subgame of Γ^e is the game induced by a $\Gamma^e|_x$ for subroot x
- an equilibrium σ of Γ^e is subgame-perfect if for each subgame of Γ^e (the obvious restriction of) σ is an equilibrium

Theorem

- if Γ^e has perfect information, then there exists at least one pure and subgame-perfect equilibrium
- any game with perfect information admits a winning strategy