# Game Theory

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## Summary of Last Lecture

### Algorithm

- INPUT: a nondegenerate bimatrix game
- OUTPUT: all Nash equilibria

### Method

1. $\forall\ k \in \{1, \dots, \min\{m, n\}\}$
2. $\forall\ k$-sized subsets $(I, J)$ of $M$, $N$
3. solve the following equation

$$\sum_{i \in I} x_i b_{ij} = v \quad \text{for } j \in J \qquad \sum_{j \in J} a_{ij} y_j = u \quad \text{for } i \in I$$

$$\sum_{i \in I} x_i = 1 \qquad \sum_{j \in J} y_j = 1$$

such that $x \geqslant \mathbf{0}$, $y \geqslant \mathbf{0}$ and the best response condition is fulfilled for $x$ and $y$





## Content







## Equilibria via Labeled Polytopes

### Definition

- an affine combination of points $z_1, \dots, z_k$ is of form

$$\sum_{i=1}^{k} z_i \lambda_i \qquad \lambda_i \in \mathbb{R} \quad \sum_{i=1}^{k} \lambda_i = 1$$

- it is called convex combination if $\lambda_i \geqslant 0$ for all $i$
- a set of points is convex if closed under forming convex combinations
- points are affinely independent if none is an affine combinations of others
- a convex set has dimension $d$ if it has $d + 1$ (but not more) affinely independent points

## Definition

- a polyhedron $P \in \mathbb{R}^d$ is a set
$$\{z \in \mathbb{R}^d \mid Cz \leqslant q\} \quad \text{for some matrix } C, \text{ vector } q$$
- $P$ is full-dimensional if it has dimension $d$ (i.e., $d+1$ (but not more) affinely independent elements)
- $P$ is a polytope if bounded
- the face of $P$ is $\{z \in P \mid c^\top z = q_0\}$ for $c \in \mathbb{R}^d$, $q_0 \in \mathbb{R}$

## Example

the following objects are polytops

- pyramids, i.e, tetrahedrons
- Rubben's cube
- octahedron
- dices in rôle games, i.e., icosahedron





## Definitionlet $P$ denote a polyhedron

- a vertex of $P$ is the unique element of a zero-dimensional face of $P$
- an edge is a one-dimensional face of $P$
- a facet of a $d$-dimensional $P$ is a $d-1$-dimensional face

## Definition — best response polyhedron

the best response polyhedron of a player is the set of that player's mixed strategies together with a bound of expected payoffs to the other player

## Example

consider $\Gamma$

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}$$

then

$$\overline{Q} = \left\{ (y_4, y_5, u) \mid \begin{array}{l} 3y_4 + 3y_5 \leqslant u,\ 3y_4 + 5y_5 \leqslant u,\ 6y_5 \leqslant u, \\ y_4 \geqslant 0,\ y_5 \geqslant 0,\ y_4 + y_5 = 1 \end{array} \right\}$$





## Definition

best response polyhedra for player 1 and 2

$$\overline{P} = \{(x,v) \in \mathbb{R}^m \times \mathbb{R} \mid x \geqslant \mathbf{0}, \mathbf{1}^\top x = 1, B^\top x \leqslant \mathbf{1}v\}$$
$$\overline{Q} = \{(y,u) \in \mathbb{R}^n \times \mathbb{R} \mid Ay \leqslant \mathbf{1}u, y \geqslant \mathbf{0}, \mathbf{1}^\top y = 1\}$$

## Definition

a point $(y,u) \in \overline{Q}$ has label $k \in M \cup N$ if

- the $k^{th}$ inequality in the definition of $\overline{Q}$ is binding
- i.e., $\sum_{j \in N} a_{kj} y_j = u$ if $k = i \in M$ or
- for $k = j \in N$, $y_j = 0$

## Example

the point $(\frac{2}{3}, \frac{1}{3}, 3)$ has labels 1 and 2, as $x_1$, $x_2$ are best responses to $y$ for player 1 that yields pay-off 3





## Definition

a point $(x,v) \in \overline{P}$ has label $k \in M \cup N$ if

- $k = i \in M$ and $x_i = 0$ or
- $k = j \in N$ and $\sum_{i \in M} b_{ik} x_i = v$

## Lemma

an equilibrium $(x,y)$ is a pair such that

- pair $((x,v),(y,u)) \in \overline{P} \times \overline{Q}$
- this pair is completely labeled, i,e. every label $k \in M \cup N$ labels either $(x,v)$ or $(y,u)$

## Proof

again this is a reformulation of the best response condition ■

## Assumptions

suppose $A$ and $B^\top$ are non-negative and have no zero columns

## Definition

consider $\overline{P}$:

- we divide each $\sum_{i \in M} b_{ij} x_i \leqslant v$ by $v$
- this gives $\sum_{i \in M} b_{ij}(\frac{x_i}{v}) \leqslant 1$
- we treat $\frac{x_i}{v}$ as a new variable (again called $x_i$)

## Definition

the normalised polytopes have the following generic form:

$$P = \{x \in \mathbb{R}^m \mid x \geqslant \mathbf{0}, B^\top x \leqslant \mathbf{1}\}$$
$$Q = \{y \in \mathbb{R}^n \mid Ay \leqslant \mathbf{1}, y \geqslant \mathbf{0}\}$$

## Lemma

the polyhedra $P$ and $Q$ are full-dimensional polytopes, moreover there is a one-to-one correspondence between $\overline{P}$ ($\overline{Q}$) and $P$ ($Q$) such that the labels are preserved





consider $\Gamma$, $A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 6 \\ 3 & 1 \end{pmatrix}$

the polyhedra $\overline{P}$, $\overline{Q}$ are defined as follows:

$$\overline{P} = \left\{ (x_1, x_2, x_3, v) \mid \begin{array}{ll} x_1 \geqslant 0 & ① \\ x_2 \geqslant 0 & ② \\ x_3 \geqslant 0 & ③ \\ 3x_1 + 2x_2 + 3x_3 \leqslant v & ④ \\ 2x_1 + 6x_2 + 1x_3 \leqslant v & ⑤ \\ x_1 + x_2 + x_3 = 1 & \end{array} \right\}$$

$$\overline{Q} = \left\{ (y_4, y_5, u) \mid \begin{array}{ll} 3y_4 + 3y_5 \leqslant u & ① \\ 3y_4 + 5y_5 \leqslant u & ② \\ 6y_5 \leqslant u & ③ \\ y_4 \geqslant 0 & ④ \\ y_5 \geqslant 0 & ⑤ \\ y_4 + y_5 = 1 & \end{array} \right\}$$





## Example

consider for example $Q$:

$$Q = \left\{ (\frac{y_4}{u}, \frac{y_5}{u}) \mid \begin{array}{ll} 3\frac{y_4}{u} + 3\frac{y_5}{u} \leqslant 1 & ① \\ 3\frac{y_4}{u} + 5\frac{y_5}{u} \leqslant 1 & ② \\ 6\frac{y_5}{u} \leqslant 1 & ③ \\ \vdots & \end{array} \right\}$$

## Observation

- $P$, $Q$ are bounded, hence polytopes
- in this transformation labels are preserved
- every vertex in $P$ ($Q$) has $m$ ($n$) labels as the game is nondegenerated





## Example

points of polytope $P$:

| | |
|---|---|
| $\mathbf{0} = (0, 0, 0)$ | labels ①, ②, ③ |
| $a = (\frac{1}{3}, 0, 0)$ | labels ②, ③, ④ |
| $b = (\frac{2}{7}, \frac{1}{14}, 0)$ | labels ③, ④, ⑤ |
| $c = (0, \frac{1}{6}, 0)$ | labels ①, ③, ⑤ |
| $d = (0, \frac{1}{8}, \frac{1}{4})$ | labels ①, ④, ⑤ |
| $e = (0, 0, \frac{1}{3})$ | labels ①, ②, ④ |

### Example (cont'd)

points of polytope $Q$:

$p = (0, \frac{1}{6})$ labels ③, ④

$q = (\frac{1}{12}, \frac{1}{6})$ labels ②, ③

$r = (\frac{1}{6}, \frac{1}{9})$ labels ①, ②

$s = (\frac{1}{3}, 0)$ labels ①, ⑤





### Algorithm

- INPUT: a nondegenerate bimatrix game
- OUTPUT: all Nash equilibria

### Method

1. define polytopes $P$, $Q$
2. $\forall$ vertex $x$ of $P - \{\mathbf{0}\}$
3. $\forall$ vertex $y$ of $Q - \{\mathbf{0}\}$
4. if $(x, y)$ is completely labeled, output the Nash equilibrium

$$(x \cdot \frac{1}{\mathbf{1}^\top x}, y \cdot \frac{1}{\mathbf{1}^\top y})$$

### Observation

vertex enumeration is more efficient than support enumeration





## Nash equilibria and NP-completeness

### Definition

a search problem $S$ consists of

1. a set of inputs $I_S \subseteq \Sigma^*$
2. $\forall\ x \in I_S\ \exists$ solution set $S_x \subseteq \Sigma^{|x|^k}$ for some integer $k$
3. such that $\forall\ x \in I_S\ \forall\ y \in \Sigma^*$ it is decidable in polytime whether $y \in S_x$

a search problem is total if $\forall\ x \in I_S\ S_x \neq \varnothing$

### Definition

we write NASH for the problem of finding a Nash equilibrium in a game in strategic form

### Example

NASH is a total search problem





## NP-completeness of Generalisations

### Definition

a bimatrix game $\Gamma$ represented by payoff matrices $A$ and $B$ is symmetric if $A = B^\top$

### Theorem

the following problems are complete for NP (even for symmetric games): given a two-player game $\Gamma$ in strategic form, does $\Gamma$ have:

- at least two Nash equilibria?
- a Nash equilibrium in which player $i$ has utility at least a given amount?
- a Nash equilibrium with support of size greater than a given number?
- a Nash equilibrium whose support contains strategy $s$?
- ...

## The Class Polynomial Parity Argument (Directed Case)

the class PPAD can be defined as the class of total search problems, where totality follows from an argument like follows

- a directed graph $G$ is defined on a finite but exponentially large set of vertexes
- each vertex has indegree and outdegree at most 1
- given a vertex $x$ is is easy to check that
  1. $x \in G$
  2. find the adjacent vertexes of $x$
  3. identify the direction of the edge
- $\exists$ a vertex with no incoming edges that is known (the standard source)
- all vertexes with no outgoing edges, or all sources other than the standard source are solutions

### Example

NASH $\in$ PPAD





## Succinct Representations of Games

### Observation ①

- given an $n$-player game
- such that each player has the same number of (pure) strategies $m$
- then representing a game in strategic form needs $nm^n$ numbers

### Observation ②

this trivialises any complexity considerations:

- the support enumeration algorithm roughly needs $(2^m)^n$ many steps
- but this is a polynomial algorithm in $nm^n$, if $m$ is fixed





## Graphical Games

### Definition

a graphical game is a $n$-person game, with $n$ large, but the utility of each player depends only on the strategies of few other players

- $\exists$ directed graph $G = (\{1, \dots, n\}, E)$
- such that $(i, j) \in E$ implies that the utility of player $j$ depends on the strategy chosen by player $i$
- $\forall$ mixed strategies $x$, $y$ if $x_j = y_j$ and $\forall$ $(i, j) \in E$: $x_i = y_i$, then $u_j(x) = u_j(y)$

### Observation ③

given a graphical $n$-player game $\Gamma$ such that

- indegree of the graph $G$ at most $d$
- maximal $m$ pure strategy per player

then $\Gamma$ needs only $nm^{d+1}$ numbers for its description





## NASH is complete for PPAD

### Definition

the problem BROUWER, a discrete version of Brouwer's fixpoint theorem: any continuous function $f$ on (let's say) cube has a fixpoint

### Theorem

NASH (even for two players) is complete for PPAD

### Proof Sketch

- BROUWER is complete for PPAD
- reduction from BROUWER to a graphical game $\Gamma$ with many players
- reduction from $\Gamma$ to NASH

### Final Remark

if P = NP, then also P = PPAD, but P = PPAD need not imply P = NP