Game Theory

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Summary of Last Lecture

Assumption

- decision makers/players act rational and intelligent

Definition

lottery

- a lottery f maps Ω to the probability distributions $\Delta(X)$

$$\sum_{x \in X} f(x|t) = 1 \qquad t \in \Omega$$

- the set of lotteries is defined as follows:

$$L = \{f \mid f\colon \Omega \to \Delta(X)\}$$

- let t be a state, $f(\cdot|t)$ denotes the probability distribution over X in t:

$$f(\cdot|t) = (f(x|t))_{x \in X} \in \Delta(X)$$

- the lottery $[x]$ always get prize $x \in X$ for sure:

$$[x](y|t) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Axiomatic Presentation ①

Axiom (totality)

- $f \succcurlyeq_S g$ or $g \succcurlyeq_S f$
- if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $f \succcurlyeq_S h$

Axiom (relevance)

$\forall t \in S$: $f(\cdot|t) = g(\cdot|t)$, then $f \sim_S g$

Axiom (monotonicity)

if $f \succ_S h$ and $\alpha > \beta$, then $\alpha f + (1-\alpha)h \succ_S \beta f + (1-\beta)h$

Axiom (continuity)

if $f \succcurlyeq_S g$ and $g \succcurlyeq_S h$, then $\exists\, \gamma\; g \sim_S \gamma f + (1-\gamma)h$

Axiom (interest)

$\forall\; t \in \Omega$, $\exists\; x, y \in X\; [y] \succ_{\{t\}} [x]$

Axiomatic Presentation ②

Axiom (objective substitution)

if $e \succcurlyeq_S f$ and $g \succcurlyeq_S h$, then
$\alpha e + (1-\alpha)g \succcurlyeq_S \alpha f + (1-\alpha)h$

Axiom (strict objective substitution)

if $e \succ_S f$ and $g \succcurlyeq_S h$, then
$\alpha e + (1-\alpha)g \succ_S \alpha f + (1-\alpha)h$

Axiom (subjective substitution)

if $f \succcurlyeq_S g$ and $f \succcurlyeq_T g$ and $S \cap T = \varnothing$, then $f \succcurlyeq_{S \cup T} g$

Axiom (strict subjective substitution)

if $f \succ_S g$ and $f \succ_T g$ and $S \cap T = \varnothing$, then $f \succ_{S \cup T} g$

Axiom (state neutrality) optional

$\forall\; r, t \in \Omega$, $f(\cdot|r) = f(\cdot|t)$, $g(\cdot|r) = g(\cdot|t)$, then $(f \succcurlyeq_{\{r\}} g) \rightarrow (f \succcurlyeq_{\{t\}} g)$

Expected Utility

recall: $\Xi = \{S \mid S \subseteq \Omega, S \neq \varnothing\}$ denotes the set of all events

Definition

- a conditional-probability function $p \colon \Xi \to \Delta(\Omega)$:

$$p(t|S) = 0 \qquad \text{if } t \notin S \qquad\qquad \sum_{r \in S} p(r|S) = 1$$

- $p(R|S) = \sum_{t \in R} p(t|S)$
- a utility function is any function from $u \colon X \times \Omega \to \mathbb{R}$
- a utility function u is state independent if $\exists\ U \colon X \to \mathbb{R}$ such that $u(x,t) = U(x)$ for all $x \in X$, $t \in \Omega$

Definition

let p denote a conditional-probability function and u any utility function, then the expected utility determined by lottery f is defined as:

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in X} u(x,t) f(x|t)$$

Content

motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, Bayesian games, Nash equilibrium

two-person zero-sum games, Bayesian equilibrium, sequential equilibria of extensive-form games, computing Nash equilibria, sub-game-perfect equilibria

efficient computation of Nash equilibria, complexity class PPAD, complexity of Nash equilibria, refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

Expected Utility Maximisation Theorem

Theorem

the axioms (with state neutrality) are satisfied if and only if there exists a (state-independent) utility function u and a conditional-probability function p such that

1. $\max_{x \in X} u(x,t) = 1$ and $\min_{x \in X} u(x,t) = 0$
2. $p(R|T) = p(R|S)p(S|T)\ \forall\ R, S, T$ so that $R \subseteq S \subseteq T$ and $S \neq \varnothing$
3. $f \succcurlyeq_S g$ if and only if $E_p(u(f)|S) \geqslant E_p(u(g)|S)$

Proof of the Theorem ①

Special Lotteries

- define a_1 for all $t \in \Omega$:

$$a_1(y|t) = 1 \quad \text{such that } \forall x \in X,\ [y] \succcurlyeq_{\{t\}} [x]$$

- define a_0 for all $t \in \Omega$:

$$a_0(y|t) = 1 \quad \text{such that } \forall x \in X,\ [x] \succcurlyeq_{\{t\}} [y]$$

More Special Lotteries

for every event $S \in \Xi$

$$b_S(\cdot|t) = \begin{cases} a_1(\cdot|t) & \text{if } t \in S \\ a_0(\cdot|t) & \text{if } t \notin S \end{cases}$$

for every event $x \in X$, $t \in \Omega$:

$$c_{x,t}(\cdot|r) = \begin{cases} [x] & \text{if } r = t \\ a_0(\cdot|r) & \text{if } r \neq t \end{cases}$$

Proof of the Theorem ②

Definition $u(x, t)$

- ask: what is the correct number β such that

$$[x] \sim_{\{t\}} \beta a_1 + (1 - \beta) a_0$$

- set $u(x, t) = \beta$

Definition $p(t|S)$

- ask: what is the correct number γ such that

$$b_{\{t\}} \sim_S \gamma a_1 + (1 - \gamma) a_0$$

- set $p(t|S) = \gamma$

Proof Plan

show that $u(x, t)$ and $p(t|S)$ fulfil the 3rd condition:

$$f \succcurlyeq_S g \Leftrightarrow E_p(u(f)|S) \geqslant E_p(u(g)|S)$$

Proof of the Theorem ③

Claim $\forall r \in \Omega$: $c_{x,t} \sim_{\{r\}} u(x, t) b_{\{t\}} + (1 - u(x, t)) a_0$

Proof

on blackboard ■

Claim $\forall S \in \Xi$: $c_{x,t} \sim_S u(x, t) b_{\{t\}} + (1 - u(x, t)) a_0$

Proof

from Claim 1 with Subjective Substitution ■

Claim

$$f \succcurlyeq_S g \Leftrightarrow \frac{1}{n} f + (1 - \frac{1}{n}) a_0 \succcurlyeq_S \frac{1}{n} g + (1 - \frac{1}{n}) a_0$$

where $n = |\Omega|$

Proof

with Objective Substitution ■

Proof of the Theorem ④

Claim

$$\frac{1}{n}f + (1 - \frac{1}{n})a_0 = \frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)c_{x,t}$$

Proof

on blackboard ∎

using the definitions of $u(x,t)$, $p(t|S)$ we have

$$\frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)c_{x,t}$$

$$\sim_S \frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)\left(u(x,t)b_{\{t\}} + (1-u(x,t))a_0\right)$$

$$\sim_S \frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)\left(u(x,t)\left[p(t|S)a_1 + (1-p(t|S))a_0\right] + \right.$$

$$\left. + (1-u(x,t))a_0\right)$$

$$\sim_S \frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)\left(u(x,t)\left[p(t|S)a_1 + (1-p(t|S))a_0\right] + \right.$$

$$\left. + (1-u(x,t))a_0\right) =$$

$$= \frac{1}{n}\sum_{t\in\Omega}\sum_{x\in X} f(x|t)u(x,t)p(t|S)a_1$$

$$+ \left(1 - \frac{1}{n}(\sum_{t\in\Omega}\sum_{x\in X} f(x|t)u(x,t)p(t|S))\right)a_0 =$$

$$= \left(\frac{E_p(u(f)|S)}{n}\right)a_1 + \left(1 - \frac{E_p(u(f)|S)}{n}\right)a_0$$

in a similar spirit, we have

$$\frac{1}{n}g + (1 - \frac{1}{n})a_0 \sim_S \left(\frac{E_p(u(g)|S)}{n}\right)a_1 + \left(1 - \frac{E_p(u(g)|S)}{n}\right)a_0$$

Proof of the Theorem ⑤

Claim

$$f \succcurlyeq_S g \Leftrightarrow$$
$$\left(\frac{E_p(u(f)|S)}{n}\right) a_1 + \left(1 - \frac{E_p(u(f)|S)}{n}\right) a_0 \succcurlyeq_S$$
$$\succcurlyeq_S \left(\frac{E_p(u(g)|S)}{n}\right) a_1 + \left(1 - \frac{E_p(u(g)|S)}{n}\right) a_0$$

Proof

by Transitivity ■

we conclude

$$f \succcurlyeq_S g \Leftrightarrow E_p(u(f)|S) \geqslant E_p(u(g)|S)$$

Proof (that 3rd property in theorem follows from axioms)

we use

- Interest and Strict Substitution to conclude $a_1 \succ_S a_0$ and
- Monotonicity ■

Violations of Strict Objective Substitution

Example

consider four lotteries

$$f_1 = 0.1 \cdot [€12\text{m}] + 0.9 \cdot [€0] \qquad f_2 = 0.11 \cdot [€1\text{m}] + 0.89 \cdot [€0]$$
$$f_3 = [€1\text{m}] \qquad f_4 = 0.10 \cdot [€12\text{m}] + 0.89 \cdot [€1\text{m}] + {} + .01 \cdot [€0]$$

Preferences

$$f_1 \succ f_2 \qquad f_3 \succ f_4$$

Observation

this violates the axiom, as

$$0.5 \cdot f_1 + 0.5 \cdot f_3 = 0.5 \cdot f_2 + 0.5 \cdot f_4$$

Example

let $X = \{-€100, €100\}$, $\Omega = \{\mathrm{L}, \mathrm{W}\}$

$$b_{\mathrm{L}}(€100|\mathrm{L}) = 1 = b_{\mathrm{L}}(-€100|\mathrm{W})$$
$$b_{\mathrm{W}}(-€100|\mathrm{L}) = 1 = b_{\mathrm{W}}(€100|\mathrm{W})$$

- L is the event where SC Austria Lustenau wins the ADEG-cup
- W is the event where Wacker Innsbruck wins
- suppose only L or W can occur

Preferences

(if someone doesn't know anything about Wacker or Lustenau)

$$0.5 \cdot [€100] + 0.5 \cdot [-€100] \succ b_{\mathrm{L}} \quad 0.5 \cdot [€100] + 0.5 \cdot [-€100] \succ b_{\mathrm{W}}$$

Observation

this violates the axiom

at least one state in Ω must have probability greater or equal than $\frac{1}{2}$

Cannot be Modelled

Example

situation A

- you buy a ticket to the movies in advance (for €10)
- on the counter you realise you've lost your ticket
- you have €10, do you buy a new ticket or go home?

situation B

- you plan to see a movie and put €10 in your pocket
- on the counter you realise you've lost your money
- do you buy a ticket with your credit card or go home?

Question

what is your preference?

$$A + \text{go home} \quad ? \quad B + \text{buy ticket}$$

Answer

no (strict) preference between A and B is modelled by the axiom: there is no difference