Game Theory

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Summary of Last Lecture

Definition

an n-person extensive-form game Γ^e is a labelled tree, where also edges are labelled such that

1. each nonterminal node has player label in $\{0, 1, \dots, n\}$
   nodes labelled with 0 are called chance nodes
   nodes labelled within $\{1, \dots, n\}$ are called decision nodes
2. edges leaving chance nodes (also called alternatives)
   are labelled with probabilities that sum up to 1
3. player nodes have a second label, the information label
   reflecting the information state
4. each alternative at a player node has a move label
5. each terminal node is labelled with $(u_1, \dots, u_n)$, the payoff

6. $\forall$ player i,
   $\forall$ nodes x y z controlled by i,
   $\forall$ alternative b at x
   - suppose y and z have the same information state
     y is reachable from x and b
   - $\exists$ node w, $\exists$ alternative c at w
     such that z follows w and c
   - and w is controlled by player i
     w has the same information label as x
     c has the same move label as b

Recall

the last assertion expresses perfect recall: whenever a player moves, she remembers all the information she knew earlier

Strategies of Players

Definition

strategy

- S_i is the set of information states per player i
- D_s is the set of possible moves at $s \in S_i$
- the set of strategies for player i is

$$\prod_{s \in S_i} D_s = \underbrace{D_s \times D_s \times \cdot \times D_s}_{|S_i|\text{-times}}$$

Example

- consider the simple card game and the strategies of player 1
- player 1 has two information states
- and each time two alternatives: Pass, Raise, or pass, raise.
- thus the set of strategies for player 1 can be represented as

$$\{(R, r), (R, p), (P, r), (P, p)\} \qquad (\text{or shorter } \{Rr, Rp, Pr, Pp\}$$

Normal Representation

given a game Γ^e in extensive form, we define the normal representation as strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$:

1. $N = \{1, \dots, n\}$, if Γ^e is an n-person game
2. for each i: C_i denotes the strategies of each player as defined above
3. we define the expected utility payoff u_i
   - set $C = \prod_{i \in N} C_i$
   - let x be a node in Γ^e
   - let $c \in C$ denote a given strategy profile
   - let $P(x|c)$ denotes the probability that the path of play goes through x, if c is chosen
   - let Ω^* denote the set of all terminal nodes
   - for $x \in \Omega^*$, $w_i(x)$ denotes the payoff for player i
   - set

$$u_i(c) = \sum_{x \in \Omega^*} P(x|c) w_i(x)$$

Content

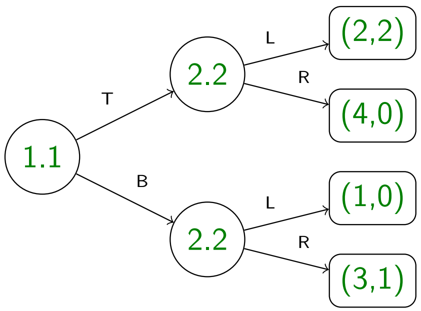
motivation, introduction to decision theory, decision theory

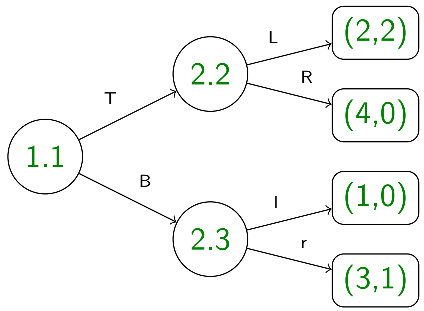
basic model of game theory, dominated strategies, Bayesian games

equilibria of strategic-form games, evolution, resistance, and risk dominance, sequential equilibria of extensive-form games, subgame-perfect equilibria, complexity of finding Nash equilibria, equilibrium computation for two-player games

refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

More Examples





C_1 \ C_2	L	R
T	2, 2	4, 0
B	1, 0	3, 1

C_1 \ C_2	Ll	Lr	Rl	Rr
T	2, 2	2, 2	4, 0	4, 0
B	1, 0	3, 1	1, 0	3, 1

Equivalence of Strategic-Form Games

Definition

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are fully equivalent if

- $\forall$ players i, $\exists$ numbers A_i and B_i
- such that $A_i > 0$
- and $u'_i(c) = A_i u_i(c) + B_i$ for any $c \in C = \prod C_i$

Example

C_1 \ C_2	x_2	y_2
x_1	9, 9	0, 8
y_1	8, 0	7, 7

C_1 \ C_2	x_2	y_2
x_1	1, 1	0, 0
y_1	0, 0	7, 7

not fully equivalent, as (x_1, x_2) is better than (y_1, y_2) in the first game, but not in the second let $C_{-i} = \prod_{j \in N \setminus \{i\}} C_j$; let (e_{-i}, d_i) denote a strategy profile, such that $e_{-i} \in C_{-i}$ and $d_i \in C_i$

- for any set Z and any $f \colon Z \to \mathbb{R}$, define

$$\text{argmax}_{y \in Z} f(y) = \{y \in Z \mid f(y) = \max_{z \in Z} f(z)\}$$

- let $\eta \in \Delta(C_{-i}) = \{q \colon C_{-i} \to \mathbb{R} \mid \sum_{e_{-i} \in C_{-i}} q(e_{-i}) = 1\}$

Definition (best response)

player i best response to η is

$$\text{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i)$$

Definition (best response equivalence)

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are best-response equivalent if (for all η)

$$\text{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i) = \text{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u'{}_i(e_{-i}, d_i)$$

Example

C_1 \ C_2	x_2	y_2
x_1	$9,9$	$0,8$
y_1	$8,0$	$7,7$

C_1 \ C_2	x_2	y_2
x_1	$1,1$	$0,0$
y_1	$0,0$	$7,7$

player 1

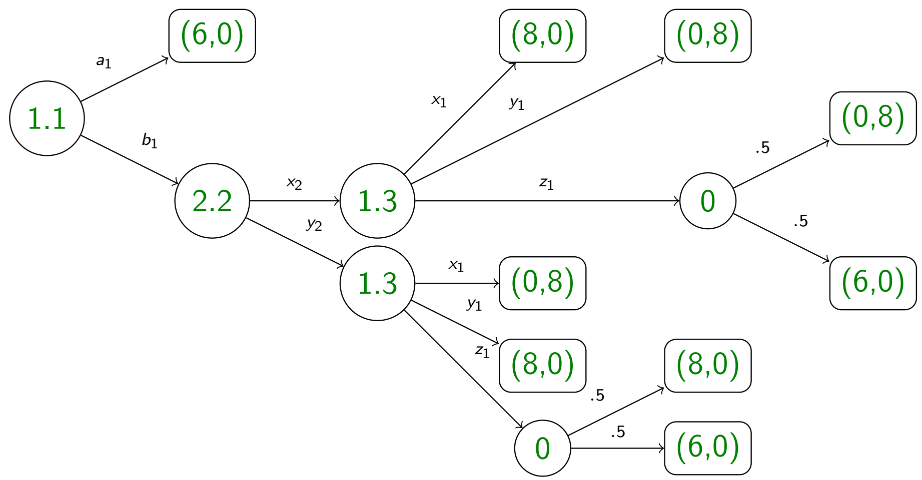
- set $\eta(x_2) = \frac{1}{2}$, $\eta(y_2) = \frac{1}{2}$
- $\text{argmax}_{d \in \{x_1, y_1\}} \frac{1}{2} u_1(d, x_2) + \frac{1}{2} u_1(d, y_2) = \text{argmax}_{d \in \{x_1, y_1\}} \frac{1}{2} u'_1(d, x_2) + \frac{1}{2} u'_1(d, y_2)$

player 2

- set $\eta(x_1) = \frac{1}{2}$, $\eta(y_1) = \frac{1}{2}$
- $\text{argmax}_{d \in \{x_2, y_2\}} \frac{1}{2} u_2(x_1, d) + \frac{1}{2} u_2(y_1, d) = \text{argmax}_{d \in \{x_2, y_2\}} \frac{1}{2} u'_1(x_1, d) + \frac{1}{2} u'_1(y_1, d)$

Example (cont'd)

the games are best-response equivalent:
as long as $\eta(y_i) \geqslant \frac{1}{8}$ the player's choose y_i, otherwise x_i



C_1 \ C_2	x_2	y_2
a_1x_1	$6,0$	$6,0$
a_1y_1	$6,0$	$6,0$
a_1z_1	$6,0$	$6,0$

C_1 \ C_2	x_2	y_2
b_1x_1	$8,0$	$0,8$
b_1y_1	$0,8$	$8,0$
b_1z_1	$3,4$	$7,0$

(Fully) Reduced Normal Representation

Definition

let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we say d_i and e_i in C_i, are payoff equivalent if

$$u_j(c_{-i}, d_i) = u_j(c_{-i}, e_i) \quad \text{for all } c_{-i} \in C_{-i}, j \in N$$

Example

strategies a_1x_1, a_1y_1, a_1z_1 are payoff equivalent

Definition (purely reduced normal representation)

identifying payoff equivalent strategies yields the purely reduced normal representation

Example

C_1 \ C_2	x_2	y_2
$a_1\cdot$	$6,0$	$6,0$
b_1x_1	$8,0$	$0,8$
b_1y_1	$0,8$	$8,0$
b_1z_1	$3,4$	$7,0$

Definition
a randomised strategy σ_i is any probability distribution over C_i (denoted $\Delta(C_i)$); i.e., $\sigma(c_i)$ denotes the probability that i choses strategy c_i

Definition
a strategey d_i is randomly redundant if $\exists\ \sigma_i \in \Delta(C_i)$ such that $\sigma_i(d_i) = 0$

$$u_j(c_{-i}, d_i) = \sum_{e_i \in C_i} \sigma_i(e_i) u_j(c_{-i}, e_i) \quad \text{for all } c_{-i} \in C_{-i}, j \in N$$

Example
consider the randomised strategy $\sigma_1 = .5[a_1\cdot] + .5[b_1 y_1]$ of player 1

- against x_2: $.5(6,0) + .5(0,8) = (3,4)$
- against y_2: $.5(6,0) + .5(8,0) = (7,0)$

strategy $a_1 z_1$ is payoff equivalent to σ_1

Definition
fully reduced normal representation

fully reduced normal representation is obtained if all randomly redundant strategies are removed

Definition
strongly dominated

let $\Gamma = (N, (C_i)_{i\in N}, (u_i)_{i \in N})$, we say d_i is strongly dominated for player i, if $\exists$ randomised strategy $\sigma_i \in \Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) > u_i(c_{-i}, d_i) \quad \text{for all } c_{-i} \in C_{-i}$$

Definition
residual game

- let $\Gamma^{(0)} = (N, (C_i)_{i\in N}, (u_i)_{i\in N}) := \Gamma$
- let $\Gamma^{(k)} = (N, (C_i^{(k)})_{i\in N}, (u_i)_{i\in N})$, such that $C_i^{(k)}$ denotes the set of all strategies in $C_i^{(k-1)}$ not strongly dominated in $\Gamma^{(k-1)}$
- clearly $C_i \supseteq C_i^{(1)} \supseteq C_i^{(2)} \supseteq \cdots \supseteq C_i^{(n)} = C_i^{(n+1)}$ as $C_i^{(n)}$ cannot become empty, but is finite
- define $\Gamma^{(\infty)} = \Gamma^{(n)}$
- the strategies $C_i^{(\infty)}$ are called iteratively undominated
- $\Gamma^{(\infty)}$ is the residual game

Example
in the card game, strategy Pp is strongly dominated by $\frac{1}{2}[Rr] + \frac{1}{2}[Rp]$

Example
consider

	C_2		
C_1	x_2	y_2	z_2
a_1	2, 3	3, 0	0, 1
b_1	0, 0	1, 6	4, 2

the residual game consists of strategy a_1 and x_2

Definition
weakly dominated

let $\Gamma = (N, (C_i)_{i\in N}, (u_i)_{i\in N})$, we say d_i is weakly dominated for player i, if $\exists$ randomised strategy $\sigma_i \in \Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \geqslant u_i(c_{-i}, d_i) \quad \text{for all } c_{-i} \in C_{-i}$$

and

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) > u_i(c_{-i}, d_i) \quad \text{for at least on } c_{-i} \in C_{-i}$$