Game Theory

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Summary of Last Lecture

Definition — fully equivalent

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are fully equivalent if

- $\forall$ players i, $\exists$ numbers A_i and B_i such that $A_i > 0$
- and $u'_i(c) = A_i u_i(c) + B_i$ for any $c \in C = \prod C_i$

$\forall\ f \colon Z \to \mathbb{R}$, define $\operatorname{argmax}_{y \in Z} f(y) = \{y \in Z \mid f(y) = \max_{z \in Z} f(z)\}$

Definition — best-response equivalence

games $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, $\Gamma' = (N, (C_i)_{i \in N}, (u'_i)_{i \in N})$ are best-response equivalent if (for all $\eta \in \Delta(C_{-i})$)

$$\operatorname{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) u_i(e_{-i}, d_i) = \operatorname{argmax}_{d_i \in C_i} \sum_{e_{-i} \in C_{-i}} \eta(e_{-i}) {u'}_i(e_{-i}, d_i)$$

Definition — strongly dominated

let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we say d_i is strongly dominated for player i, if $\exists$ randomised strategey $\sigma_i \in \Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) > u_i(c_{-i}, d_i) \quad \text{for all } c_{-i} \in C_{-i}$$

Definition — weakly dominated

let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we say d_i is weakly dominated for player i, if $\exists$ randomised strategy $\sigma_i \in \Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \geqslant u_i(c_{-i}, d_i) \qquad \text{for all } c_{-i} \in C_{-i}$$

and

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) > u_i(c_{-i}, d_i) \qquad \text{for at least on } c_{-i} \in C_{-i}$$

Elimination of Dominated Strategies

Definition — residual game

- let $\Gamma^{(0)} = (N, (C_i)_{i \in N}, (u_i)_{i \in N}) := \Gamma$
- let $\Gamma^{(k)} = (N, (C_i^{(k)})_{i \in N}, (u_i)_{i \in N})$, such that $C_i^{(k)}$ denotes the set of all strategies in $C_i^{(k-1)}$ not strongly dominated in $\Gamma^{(k-1)}$
- clearly $C_i \supseteq C_i^{(1)} \supseteq C_i^{(2)} \supseteq \cdots \supseteq C_i^{(n)} = C_i^{(n+1)}$ as $C_i^{(n)}$ cannot become empty, but is finite
- define $\Gamma^{(\infty)} = \Gamma^{(n)}$
- the strategies $C_i^{(\infty)}$ are called iteratively undominated
- $\Gamma^{(\infty)}$ is the residual game

Content

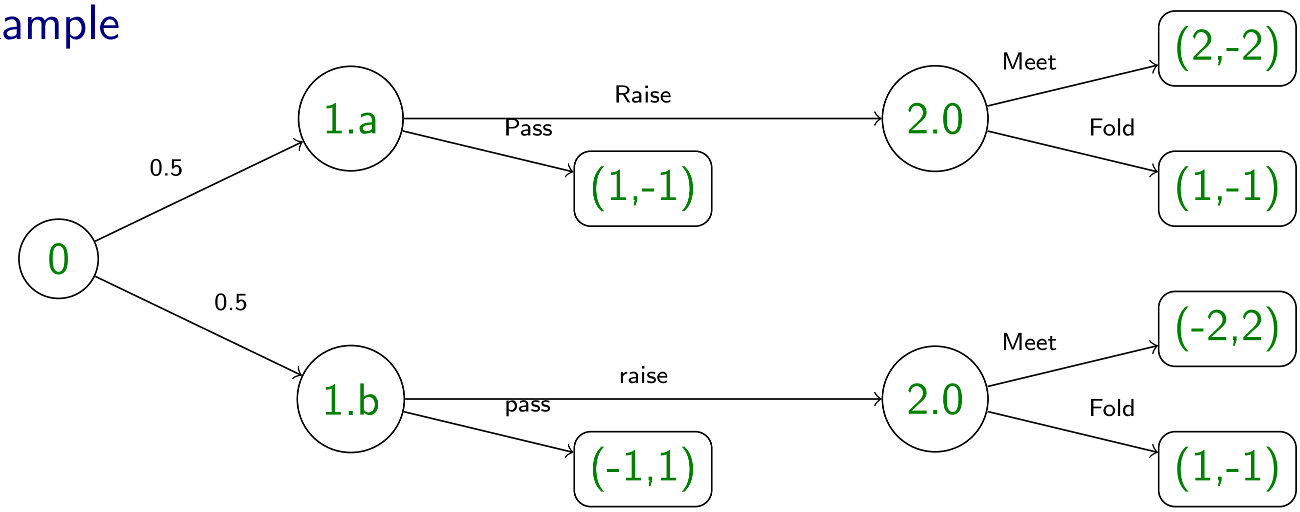
motivation, introduction to decision theory, decision theory

basic model of game theory, dominated strategies, common knowledge, Bayesian games, incomplete information, Nash equilibrium

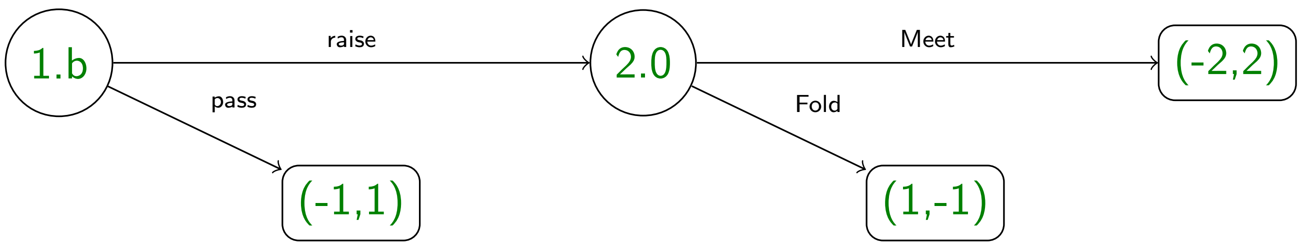
two-person zero-sum games, Bayesian equilibria, sequential equilibra of extensive-form games, computing Nash equilibria, subgame-perfect equilibra

efficient computation of Nash equilibria, complexity class PPAD, complexity of Nash equilibria, refinements of equilibrium in strategic form, persistent equilibria, games with communication, sender-receiver games

Example



suppose player 1 has drawn a black card; consultant models game as follows



Common Knowledge is Important

Definition

common knowledge

- common knowledge among the players holds, if every player knows it, every player knows that every player knows it, and so on
  the statement (every player knows it)^k is true for all $k \geqslant 0$
- private information is any information of a player, that is not common knowledge

Example

- two red-hat smurfs and two blue-hat smurfs travel around logic land, where they become prisoners to an evil logician
- they are placed before and after a wall and there hats get exchanged as follows

  R | B R B

Question

which smurf can deduct the colour of his (or her) hat?

Example

- in logic land there is a region where 100 couples live
- every night all men meet and either praise their wifes or curse them
- they praise their wifes if they cannot conclude that they have been unfaithful
- otherwise they curse them
- whenever a woman is unfaithul, she and her lover inform everybody, except the husband

Facts

- for ages all the men praised their wifes
- but actually all the women have been unfaithful

A Stranger Enters

- one day a stranger announces that $\exists$ an unfaithful wife
- for 99 day all the men continue to praise their wifes
- on the 100th day, the start to curse, moan and wail

Question

why?

Answer

- every man knew of 99 unfaithful wives
- but not that his own wife was unfaithful
- so "(every man knows that)^k there is an unfaithful wife" for $k \leqslant 99$
- so 1 knew that 2 knew that 3 knew ... that 99 knew that 100's wife was unfaithful
- after the stranger speaks (and some time) the cylce closes ∎

reasoning about common knowledge can be formalised using modal and fixed-point logic

Definition — incomplete information

- a game has incomplete information if some players have private information before the game starts
- the initial private information is called the type of the player

Definition — Bayesian games

a Bayesian game is a tuple $\Gamma^b = (N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N})$ such that

1. N is the set of players
2. C_i is the set of actions of player i
3. T_i is the set of types of player i
4. set $C = \prod_{i \in N} C_i$, $T = \prod_{i \in N} T_i$
5. $p_i(\cdot|t_i) \in \Delta(T_{-i})$ is the probability distribution over the types of the other players T_{-1}
6. for each i: $u_i \colon C \times T \to \mathbb{R}$ is the expected utility payoff

Definition

a strategy for player i in Γ^b is a function $f\colon T \to C$

Example

consider the card game with the alteration that player 1 already knows the colour of the card

$$\Gamma^b = (\{1,2\}, C_1, C_2, T_1, T_2, p_1, p_2, u_1, u_2)$$

- $C_1 = \{R, P\},\ C_2 = \{M, F\}$
- $T_1 = \{1.a, 1.b\},\ T_2 = \{2\}$
- $p_1(2|1.a) = p_1(2|1.b) = 1,\ p_2(1.a|2) = p_2(1.b|2) = 0.5$
- the utility functions depend on (c_1, c_2, t_1) as follows:

$t_1 = 1.a$	M	F
R	2,-2	1,-1
P	1,-1	1,-1

$t_1 = 1.b$	M	F
R	-2,2	1,-1
P	-1,1	-1,1

Example

consider bargaining game: player 1 is the seller, player two is the buyer

- each player knows the value of the object to himself; assumes the value to the other is $\in [1, 100]$ with uniform probability
- each player bids a number $\in [0, 100]$
- assume utility $=$ monetary profit

Formalisation

$\Gamma^b = (\{1,2\}, C_1, C_2, T_1, T_2, p_1, p_2, u_1, u_2)$ such that

1. $C_1 = C_2 = [0, 100],\ T_1 = T_2 = [1, 100]$
2. $\forall i \in N, \forall t = (t_{-i}, t_i) \in T\ \ p_i(t_{-i}|t_i) = \frac{1}{100}$
3. $u_1(c, t) = \frac{c_1 + c_2}{2} - t_1$ if $c_2 \geqslant c_1$
4. $u_2(c, t) = t_2 - \frac{c_1 + c_2}{2}$ if $c_2 \geqslant c_1$
5. $u_1(c, t) = u_2(c, t) = 0$ if $c_2 < c_1$

$\forall\ c \in C,\ t \in T$

Observation

- it may easier to analyse games with infinite type sets than games with large finite sets of types
- in the infinite case it suffices to define $p_i(\cdot|t_i)$ on all (measurable) subsets of T_{-i}

Example (cont'd)

$$p_i([x,y]|t_i) = \frac{(y-x)}{100}$$

Definition

a set of beliefs $(p_i)_{i \in N}$ in a Bayesian game is consistent if there exists a probability distribution $P \in \Delta(T)$ such that

$$p_i(t_{-i}|t_i) = \frac{P(t)}{\sum_{s_{-i} \in T_{-i}} P(s_{-i}, t_i)} \qquad \forall\ t \in T,\ i \in N$$

any Bayesian game is representable as strategic game by conceiving each type as a player

Beyond Domination

Example

consider the normal representation of the card game

C_1	C_2: M	F
Rr	$0, 0$	$1, -1$
Rp	$0.5, -0.5$	$0, 0$
Pr	$-0.5, 0.5$	$1, -1$
Pp	$0, 0$	$0, 0$

Question

can we exclude strategy Pr?

Answer

not yet, as it is only weakly dominated, but not strongly let $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$

- a randomised strategy for player i,
  is a probability distribution $\Delta(C_i)$ over C_i
- $c_i \in C_i$ is a pure strategy
- a randomised strategy profile $\sigma \in \prod_{i \in N} \Delta(C_i)$ specifies a randomised strategy for every player

Definition

let $\sigma \in \prod_{i \in N} \Delta(C_i)$, let $u_i(\sigma)$ denote the expected utility payoff for player i, when players choose strategies according to σ:

$$u_i(\sigma) = \sum_{c \in C} \Big(\prod_{j \in N} \sigma_j(c_j)\Big) u_i(c) \quad \text{for all } i \in N$$

for $\tau_i \in \Delta(C_i)$, let (σ_{-i}, τ_i) denote the randomised strategy profile, where τ_i is substituted for σ_i, thus

$$u_i(\sigma_{-i}, \tau_i) = \sum_{c \in C} \Big(\prod_{j \in N \setminus \{i\}} \sigma_j(c_j)\Big) \tau_i(c_i) u_i(c)$$

Nash Equilibrium

let $[c_i] \in \Delta(C_i)$ such that

$$[c_i](x) = \begin{cases} 1 & x = c_i \\ 0 & \text{otherwise} \end{cases}$$

Notation

if player i uses d_i, while all other players behave independently according to $\sigma_i \in \prod_{i \in N} \Delta(C_i)$, we have

$$u_i(s_{-1}, [d_i]) = \sum_{c_{-1} \in C_{-1}} \Big(\prod_{j \in N \setminus \{i\}} \sigma_j(c_j)\Big) u_i(c_{-i}, d_i)$$

Definition

Nash equilibrium

a randomised strategy profile σ is a (mixed) Nash equilibrium of Γ if the following holds for all $i \in N$, and every $c_i \in C_i$

$$\text{if } \sigma_i(c_i) > 0, \text{ then } c_i \in \text{argmax}_{d_i \in C_i} u_i(\sigma_{-i}, [d_i])$$

Lemma

- for any $\sigma \in \prod_{i \in N} \Delta(C_i)$ and any player i

$$\max_{c_i \in C_i} u_i(\sigma_{-i}, [c_i]) = \max_{\tau_i \in \Delta(C_i)} u_i(\sigma_{-i}, \tau_i)$$

- furthermore, $p_i \in \text{argmax}_{\tau_i \in \Delta(C_i)} u_i(\sigma_{-i}, \tau_i)$ if and only if $p_i(c_i) = 0$ for every $c_i \notin \text{argmax}_{c_i \in C_i} u_i(\sigma_{-i}, c_i)$

the highest expected utility player i can get is independent of the fact whether player i used randomised strategies for herself

a pure strategy profile $c \in C$ is a pure Nash equilibrium if for all $i \in N$, and every $d_i \in C_i$

$$u_i(c) \geqslant u_i(c_{-i}, d_i)$$

Example

Example

consider the following game

	C_2		
C_1	x_2	y_2	z_2
x_1	3,0	0,2	0,3
y_1	2,0	1,1	2,0
z_1	0,3	0,2	3,0

the unique Nash equilibrium is (y_1, y_2)

Observation

- none of the strategies are (weakly, strongly) dominated
- every strategy is best response to one of the other player's strategies