1. Consider the following sentences:

   ① A troll is yellow if its father or mother is yellow.
   ② A troll can perform magic if all its relatives can perform magic.
   ③ Trolls only stink if they are blue or bathing in mud.
   ④ Blue trolls eat spiders if they do not eat worms.
   ⑤ Xibu is a yellow troll who cannot perform magic.

   a) For each of the sentences above, give a first-order formula that formalises the sentence. Use therefore the following constants, functions and predicates:
      - constants: mud, worms, spiders, Xibu
      - functions: father(x), mother(x)
      - predicates: Troll(x), Yellow(x), Blue(x), Magic(x), Stink(x), Relative(x, y), Bathing(x, y), Eat(x, y)

      Note that the predicate Magic(x) are to be interpreted as "x performs magic", the predicate Relative(x, y) as "x is a relative of y", the predicate Bathing(x, y) as "x is bathing in y" and the predicate Eat(x, y) as "x eats y". (5 pts)

   b) Show that your formalisation is satisfiable. (3 pts)

2. Consider the following attempt of a definition:

   Wrong Definition. An *interpretation* $\mathcal{I}$ is a structure $\mathcal{A}$ and the *value* of a term t (possible containing free variables) with respect to $\mathcal{I}$ is defined as follows:

   $$t^{\mathcal{I}} := f^{\mathcal{A}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) \qquad \text{if } t = f(t_1, \dots, t_n) \; .$$

   a) Give an example, where this definition is ill-defined. (4 pts)

   b) Correct the definition. (4 pts)

3. Consider the following sentences in prenex normal form:
   - $F_1 :\Longleftrightarrow \forall x \exists y \forall z \forall u \exists w (Q(x, y, z) \rightarrow P(w, x, y, u))$.
   - $F_2 :\Longleftrightarrow \exists x \forall y \forall z \exists w (R(x, z) \wedge R(x, y) \rightarrow (R(x, w) \wedge R(y, w) \wedge R(z, w)))$.
   - $F_3 :\Longleftrightarrow \forall x \forall y \exists z \exists u \exists v (S(y, z) \wedge (S(z, u) \wedge (S(x, v) \wedge S(v, u))))$.

   a) Define the SNFs G_i ($i = 1, 2, 3$) of the sentences given above. (6 pts)

   b) Consider a satisfiable set $\mathcal{G}$ of sentences (not containing $=$). Give two consequences according to Herbrand's Theorem. (You may use any notions introduced in the lecture, but if you define operators please shortly indicate their definitions.) (4 pts)

   c) Let $\mathcal{L} = \{\mathsf{c}, \mathsf{P}\}$. Give an example of a sentence F over $\mathcal{L}$ involving quantifiers and a finite Herbrand model of F. (4 pts)

4. Consider the formula (predicate constants P, Q):

$$(\forall x(P(x) \lor Q(x))) \to (\exists x P(x) \lor \forall x Q(x)) \ .$$

a) Is this formula valid or not? (2 pts)

b) If the formula is valid, provide evidence of this fact: either give a semantic argument, a natural deduction proof, or a resolution proof. Otherwise, give a suitable counter-model. (8 pts)

5. Determine whether the statements on the answer sheet are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

- Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that the respective universes coincide. Then for any formula F: $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$.
- Let $\mathcal{A}$, $\mathcal{B}$ be structures and $\mathcal{A} \cong \mathcal{B}$. Then for every sentence F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
- Suppose $\mathcal{G}$ is a set of formulas and $\mathcal{G} \models F$. Then there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{G}_0 \models F$.
- If a set of formulas $\mathcal{G}$ (over a language containing $=$) has a model, then $\mathcal{G}$ also has a countable infinite model.
- Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations, such that $\mathcal{I}_2$ is a subinterpretation of $\mathcal{I}_1$. If F is a universal sentence and $\mathcal{I}_1 \models F$, then $\mathcal{I}_2 \models F$.
- There exists a satisfiable set of sentences $\mathcal{G}$, such that there exists no Herbrand model of $\mathcal{G}$.
- Suppose the sentence $A \to C$ is valid. Then there exists no sentence B such that $A \to B$ and $B \to C$ are valid.
- Second-order logic is neither complete, compact, nor satisfies Löwenheim-Skolem.
- Reachability in directed graphs is expressible as existential second-order formula.
- For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $F \approx \forall x_1 \ldots \forall x_n (C_1 \land \cdots \land C_m)$.