1. a) ① $\forall x\ (\mathsf{Troll}(x) \wedge [\mathsf{Yellow}(\mathsf{father}(x)) \vee \mathsf{Yellow}(\mathsf{mother}(x))] \rightarrow \mathsf{Yellow}(x))$
   ② $\forall x\ (\mathsf{Troll}(x) \wedge \forall y\ [\mathsf{Relative}(y, x) \rightarrow \mathsf{Magic}(y)] \rightarrow \mathsf{Magic}(x))$
   ③ $\forall x\ (\mathsf{Troll}(x) \rightarrow [\mathsf{Stink}(x) \rightarrow (\mathsf{Blue}(x) \vee \mathsf{Bathing}(x, \mathsf{Mud}))])$
   ④ $\forall x\ (\mathsf{Troll}(x) \wedge \mathsf{Blue}(x) \rightarrow [\neg\mathsf{Eat}(x, \mathsf{worms}) \rightarrow \mathsf{Eat}(x, \mathsf{spiders})])$
   ⑤ $\mathsf{Troll}(\mathsf{Xibu}) \wedge \mathsf{Yellow}(\mathsf{Xibu}) \wedge \neg\mathsf{Magic}(\mathsf{Xibu})$

   b) We define a suitable structure $\mathcal{A} = (A, a)$ where $A = \{m, w, s, xibu\}$ and the mapping a is defined as follows:
   - $a(\mathsf{mud}) = m$, $a(\mathsf{worms}) = w$, $a(\mathsf{spiders}) = s$, $a(\mathsf{Xibu}) = xibu$.
   - $a(\mathsf{father}) = a(\mathsf{mother}) := f\colon \{m, w, s, xibu\} \mapsto m$, that is all function symbols are mapped to the constant function that always returns m.
   - $a(\mathsf{Troll}) = a(\mathsf{Yellow}) = \{xibu\}$, $a(\mathsf{Relative}) = \{(xibu, xibu)\}$ and $a(\mathsf{Blue}) = a(\mathsf{Magic}) = a(\mathsf{Stink}) = a(\mathsf{Bathing}) = a(\mathsf{Eat}) = \varnothing$.

   Let l be an arbitrary assignment and $\mathcal{M} = (\mathcal{A}, l)$. Then $\mathcal{M} \models F$ holds for each of the five sentences above and thus the formalisation is satisfiable.

2. The simplest example of a term t, where the definition is ill-defined would be $t = a$, a a variable. Definition 1.7 on page 4 in the lecture notes gives the correct definition.

3. Consider the following sentences in SNF:

   a)
   - $F_1^S :\Longleftrightarrow \forall x \forall z \forall u (\neg Q(x, f(x), z) \vee P(g(x, z, u), x, f(x), u))$
   - $F_2^S :\Longleftrightarrow \forall y \forall z ((\neg R(a, z) \vee \neg R(a, y) \vee R(a, f(y, z))) \wedge (\neg R(a, z) \vee \neg R(a, y) \vee R(y, f(y, z))) \wedge (\neg R(a, z) \vee \neg R(a, y) \vee R(z, f(y, z))))$
   - $F_3^S :\Longleftrightarrow \forall x \forall y (S(y, f(x, y)) \wedge S(f(x, y), g(x, y)) \wedge S(x, h(x, y)) \wedge S(h(x, y), g(x, y)))$

   where a, f, g, h are newly introduced Skolem constants or Skolem functions, respectively.

   b) See Theorem 5.4 or Corollary 5.2 on page 40ff in the lecture notes and note that these consequences only hold for *universal* sentences.

   c) Consider the sentence $F := \exists x \mathsf{P}(x)$. In order to define an Herbrand model $\mathcal{M}$ consider the finite Herband universe $H = \{c\}$ together with the interpretation $c^{\mathcal{M}} := c$. To make $\mathcal{M}$ a model of F it suffices to set $\mathsf{P}^{\mathcal{M}}(c)$ true in $\mathcal{M}$.

   (Note that F does not contain any function symbols or constants. In this case we may add an arbitrary fresh constant to the Herband universe H.)

4. a) The formula is valid.

   b) Use ordered resolution, where we assume $P(t) \succ Q(t)$ for all ground terms t and $\succ$ is lifted to a order on literals as in the lecture.

   Let F denote the formula. Then we obtain for the SNF of $\neg F$:

$$\forall x \forall y\, ((P(x) \vee Q(x)) \wedge \neg P(y) \wedge \neg Q(f(x, y)) \ ,$$

where f is a new Skolem function. Let $\mathcal{C} = \{P(x) \vee Q(x), \neg P(x), \neg Q(f(x,y))\}$ denote the corresponding set of clauses.

The (ordered) resolution proof is given below, where for each inference σ denotes the most general unifier.

$$\dfrac{\dfrac{P(x) \vee Q(x) \quad \neg P(x')}{Q(x)}\ \sigma = \{x' \mapsto x\} \qquad \neg Q(f(x',y'))}{\square}\ \sigma = \{x \mapsto f(x',y')\}$$

5.

statement	yes	no
Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that the respective universes coincide. Then for any formula F: $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$.	☐	✓
Let $\mathcal{A}$, $\mathcal{B}$ be structures and $\mathcal{A} \cong \mathcal{B}$. Then for every sentence F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.	✓	☐
Suppose $\mathcal{G}$ is a set of formulas and $\mathcal{G} \models F$. Then there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{G}_0 \models F$.	✓	☐
If a set of formulas $\mathcal{G}$ (over a language containing =) has a model, then $\mathcal{G}$ also has a countable infinite model.	☐	✓
Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations, such that $\mathcal{I}_2$ is a subinterpretation of $\mathcal{I}_1$. If F is a universal sentence and $\mathcal{I}_1 \models F$, then $\mathcal{I}_2 \models F$.	✓	☐
There exists a satisfiable set of sentences $\mathcal{G}$, such that there exists no Herbrand model of $\mathcal{G}$.	✓	☐
Suppose the sentence $A \to C$ is valid. Then there exists no sentence B such that $A \to B$ and $B \to C$ are valid.	☐	✓
Second-order logic is neither complete, compact, nor satisfies Löwenheim-Skolem.	✓	☐
Reachability in directed graphs is expressible as existential second-order formula.	✓	☐
For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $F \approx \forall x_1 \ldots \forall x_n (C_1 \wedge \cdots \wedge C_m)$.	✓	☐