1. Consider the following sentences:

   ① Each dragon is happy if all its children are happy.
   ② Dragons can fly if and only if at least one of their ancestors can fly.
   ③ A dragon is green if one of its parents is red.
   ④ Green dragons cannot spit fire.
   ⑤ There are red dragons that cannot fly.

   a) For each of the sentences above, give a first-order formula that formalises it. Use (only) the following constants, functions and predicates:
      - constants: green, red.
      - functions: $\mathsf{colour}(x)$.
      - predicates: $\mathsf{Dragon}(x)$, $\mathsf{Happy}(x)$, $\mathsf{Fly}(x)$, $\mathsf{Child}(x, y)$, $\mathsf{Ancestor}(x, y)$, $\mathsf{Spitfire}(x)$, $=$.

      Note that the predicate $\mathsf{Child}(x, y)$ is to be interpreted as "x is a child of y" and the predicate $\mathsf{Ancestor}(x, y)$ as "x is a ancestor of y". (5 pts)

   b) Show that your formalisation is satisfiable. (3 pts)

2. Consider the following ill-defined definition.

   Wrong Definition. Let $\mathcal{A}$, $\mathcal{B}$ be two structures (with respect to the same language $\mathcal{L}$) and let A, B denote the respective domains. Suppose there exists a bijection $m \colon A \to B$ such that

   a) for any individual constant c, $m(c^{\mathcal{A}}) = c^{\mathcal{B}}$,
   b) for any n-ary function constant f and all $a_1, \ldots, a_n \in A$ we have

   $$m(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(m(a_1), \ldots, m(a_n)) \text{ , and}$$

   then m is called an *isomorphism*. We write $\mathcal{A} \cong_1 \mathcal{B}$ if there exists an isomorphism $m \colon \mathcal{A} \to \mathcal{B}$.

   a) The definition is wrong, correct it. (5 pts)
   b) Let $\mathcal{A}$, $\mathcal{B}$ be structures such that $\mathcal{A} \cong \mathcal{B}$. Then we have: $\mathcal{A} \models F$ iff $\mathcal{B} \models F$. Give counter-examples if the (ill-defined) relation $\cong_1$ is used instead of $\cong$. (5 pts)

3. Consider the following sentences in prenex normal form:

   - $F_1 :\Longleftrightarrow \forall x \forall y (x < y \to \exists z (x < z \land z < y))$
   - $F_2 :\Longleftrightarrow \forall x (\exists y \mathsf{P}(y) \to \mathsf{P}(x))$
   - $F_3 :\Longleftrightarrow \forall x (\mathsf{Q}(x) \to \exists y (\mathsf{P}(y) \land \mathsf{R}(y, x))) \to \exists x \mathsf{S}(x)$

   a) Define the SNFs G_i $(i = 1, 2, 3)$ of the sentences F_i given above. (6 pts)

b) Consider the following claim: *For any formula F and its SNF G we have $F \equiv G$.* Decide whether this claim is correct, and explain your answer. (4 pts)

c) Let $\mathcal{L} = \{\mathsf{c}, \mathsf{f}, \mathsf{P}\}$. Consider the sentence $G :\Longleftrightarrow \mathsf{P}(\mathsf{c}) \wedge \forall x(\mathsf{P}(x) \rightarrow \mathsf{P}(\mathsf{f}(x))) \wedge \exists x \neg \mathsf{P}(x)$. Extend $\mathcal{L}$ to a language $\mathcal{L}'$ such that there exists a Herbrand model $\mathcal{I}$ (of $\mathcal{L}'$) of G. (3 pts)

4. Consider the following set of clauses $\mathcal{C}$ (individual constants a, b, predicate constants $\mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}$):

$$\{\mathsf{P}(x) \vee \mathsf{Q}(x) \vee \mathsf{R}(x, y), \neg \mathsf{P}(x), \neg \mathsf{Q}(\mathsf{a}), \mathsf{S}(\mathsf{a}, y) \vee \neg \mathsf{R}(\mathsf{a}, y) \vee \mathsf{S}(x, \mathsf{b}), \neg \mathsf{S}(\mathsf{a}, \mathsf{b}) \vee \neg \mathsf{R}(\mathsf{a}, \mathsf{b})\}$$

a) Is $\mathcal{C}$ satisfiable or not? (2 pts)

b) If $\mathcal{C}$ is satisfiable, give a model $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{C}$ otherwise, give an ordered resolution proof to verify this. You may assume the following relations on ground atoms and lift $\succ$ to a order on literals as in the lecture.

$$\mathsf{P}(t_1) \succ \mathsf{Q}(t_2) \succ \mathsf{S}(t_3, t_4) \succ \mathsf{R}(t_5, t_6) ,$$

for any ground terms $t_1, \ldots, t_6$. (7 pts)

5. Determine whether the statements on the answer sheet are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

- Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that the respective universes coincide and suppose $\mathcal{I}_1, \mathcal{I}_2$ coincide on the constants in the closed formula F. Then $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$.
- For all formulas F and all sets of formulas $\mathcal{G}$ we have that $\mathcal{G} \models F$ iff $\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$.
- Let A, B be sets such that there exists a bijection m between them. Then if $\mathcal{A}$ is a structure with domain A, there exists a structure $\mathcal{B}$ with domain B such that $\mathcal{A} \cong \mathcal{B}$.
- If there exists a finite subset of a set of formulas $\mathcal{G}$ that has a model, then $\mathcal{G}$ has a model.
- If a set of formulas $\mathcal{G}$ has an infinite model, then $\mathcal{G}$ also has a countable infinite model.
- If the sentence $A \rightarrow C$ holds, then there exists a sentence B such that $A \rightarrow B$ and $B \rightarrow C$.
- Second-order logic is neither complete, compact, nor satisfies Löwenheim-Skolem.
- Reachability in directed graphs is expressible as existential, first-order formula.
- It is undecidable whether two given terms s, t are unifiable.
- For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $F \approx \forall x_1 \ldots \forall x_n (C_1 \wedge \cdots \wedge C_m)$.