1. Consider the following sentences:

   ① Each smurf is happy if all its children are happy.
   ② Smurfs are green if at least two of their ancestors are green.
   ③ A smurf is really small if one of its parents is large.
   ④ Large smurfs are not really small.
   ⑤ There are red smurfs that are large.

   a) For each of the sentences above, give a first-order formula that formalises it. Use *only* the following constants, functions and predicates:
      - constants: green, red.
      - functions: colour(x).
      - predicates: Smurf(x), Large(x), ReallySmall(x), Happy(x), Child(x, y), Ancestor(x, y), $=$. (5 pts)

   b) Show that your formalisation is satisfiable. (3 pts)

2. Consider the following attempt of a definition:

   Wrong Definition. An *interpretation* $\mathcal{I}$ is a structure $\mathcal{A}$ and the *value* of a term t (possible containing free variables) with respect to $\mathcal{I}$ is defined as follows:

   $$t^{\mathcal{I}} := f^{\mathcal{A}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}}) \qquad \text{if } t = f(t_1, \ldots, t_n) \ .$$

   a) Give an example, where this definition is ill-defined. (5 pts)

   b) Correct the definition. (5 pts)

3. Consider the following sentences in prenex normal form:
   - $F_1 :\Longleftrightarrow \forall y \forall x (x > y \rightarrow \exists z (x > z \land z > y))$
   - $F_2 :\Longleftrightarrow \forall x (\exists y \mathsf{Q}(y) \rightarrow \mathsf{P}(x))$
   - $F_3 :\Longleftrightarrow \forall x (\mathsf{P}(x) \rightarrow \exists y (\mathsf{Q}(y) \lor \mathsf{R}(y, x))) \rightarrow \exists x \mathsf{S}(x)$

   a) Define the SNFs G_i ($i = 1, 2, 3$) of the sentences F_i given above. (6 pts)

   b) Consider the following claim: *For any formula F and its SNF G we have $F \equiv G$.* Decide whether this claim is correct, and explain your answer. (4 pts)

   c) Let $\mathcal{L} = \{\mathsf{c}, \mathsf{f}, \mathsf{P}\}$. Consider the sentence $G :\Longleftrightarrow \mathsf{P}(\mathsf{c}) \land \forall x (\mathsf{P}(x) \rightarrow \mathsf{P}(\mathsf{f}(x))) \land \exists x \neg \mathsf{P}(x)$. Extend $\mathcal{L}$ to a language $\mathcal{L}'$ such that there exists a Herbrand model $\mathcal{I}$ (of $\mathcal{L}'$) of G. (3 pts)

4. Consider the following set of clauses $\mathcal{C}$ (individual constants a, b, predicate constants $\mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}$):

   $$\{\mathsf{P}(x) \lor \mathsf{Q}(x) \lor \mathsf{R}(x, y), \neg \mathsf{P}(x), \neg \mathsf{Q}(\mathsf{a}), \mathsf{S}(\mathsf{a}, y) \lor \neg \mathsf{R}(\mathsf{a}, y) \lor \mathsf{S}(x, \mathsf{b}), \neg \mathsf{S}(\mathsf{a}, \mathsf{b}) \lor \neg \mathsf{R}(\mathsf{a}, \mathsf{b})\}$$

a) Is $\mathcal{C}$ satisfiable or not? (2 pts)

b) If $\mathcal{C}$ is satisfiable, give a model $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{C}$ otherwise, give an ordered resolution proof to verify this. You may assume the following relations on ground atoms and lift $\succ$ to a order on literals as in the lecture.

$$\mathsf{P}(t_1) \succ \mathsf{Q}(t_2) \succ \mathsf{S}(t_3, t_4) \succ \mathsf{R}(t_5, t_6) \ ,$$

for any ground terms $t_1, \dots, t_6$. (7 pts)

5. Determine whether the statements on the answer sheet are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

- Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that the respective universes coincide and suppose $\mathcal{I}_1, \mathcal{I}_2$ coincide on the constants in the closed formula F. Then $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$.
- Let $\mathcal{A}$, $\mathcal{B}$ be structures and $\mathcal{A} \cong \mathcal{B}$. Then for every sentence F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
- For all formulas F and all sets of formulas $\mathcal{G}$ we have that $\mathcal{G} \models F$ iff $\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$.
- Suppose $\mathcal{G}$ is a set of formulas and $\mathcal{G} \models F$. Then there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{G}_0 \models F$.
- Let A, B be sets such that there exists a bijection m between them. Then if $\mathcal{A}$ is a structure with domain A, there exists a structure $\mathcal{B}$ with domain B such that $\mathcal{A} \cong \mathcal{B}$.
- Suppose the sentence $A \to C$ is valid. Then there exists no sentence B such that $A \to B$ and $B \to C$ are valid.
- If a set of formulas $\mathcal{G}$ has an infinite model, then $\mathcal{G}$ has no countable infinite model.
- If the sentence $A \to C$ holds, then there exists a sentence B such that $A \to B$ and $B \to C$.
- For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \dots, C_m\}$ such that $F \approx \forall x_1 \dots \forall x_n (C_1 \wedge \cdots \wedge C_m)$.
- Reachability in directed graphs is expressible as a second-order formula.