1. a) ① $\forall x\ (\mathsf{Smurf}(x) \land \forall y\ (\mathsf{Child}(y,x) \rightarrow \mathsf{Happy}(y)) \rightarrow \mathsf{Happy}(x))$.
   ② $\forall x\ (\mathsf{Smurf}(x) \land \exists y_1 \exists y_2 (\mathsf{Ancestor}(y_1,x) \land \mathsf{Ancestor}(y_2,x) \land y_1 \neq y_2 \land \mathsf{colour}(y_1) = \mathsf{green} \land \mathsf{colour}(y_1) = \mathsf{green}) \rightarrow \mathsf{colour}(x) = \mathsf{green})$.
   ③ $\forall x\ (\mathsf{Smurf}(x) \land \exists y (\mathsf{Child}(x,y) \land \mathsf{Large}(y)) \rightarrow \mathsf{ReallySmall}(x))$.
   ④ $\forall x\ (\mathsf{Smurf}(x) \land \mathsf{Large}(x) \rightarrow \neg\mathsf{ReallySmall}(x))$.
   ⑤ $\exists x\ (\mathsf{Smurf}(x) \land (\mathsf{color}(x) = \mathsf{red}) \land \mathsf{Large}(x))$.

   b) We define a suitable structure $\mathcal{A} = (A, a)$ where

   $$A = \{green, red, schlumpfine, gargamel\}\ ,$$

   and the mapping a is defined as follows:
   - $\mathsf{green}^{\mathcal{A}} = green$, $\mathsf{red}^{\mathcal{A}} = red$.
   - $\mathsf{color}^{\mathcal{A}} = f\colon \{green, red, schlumpfine, gargamel\} \mapsto red$, that is all function symbols are mapped to the constant function that always returns red.
   - $\mathsf{Smurf}^{\mathcal{A}} = \mathsf{Large}^{\mathcal{A}} = \mathsf{Happy}^{\mathcal{A}} = \{schlumpfine\}$ and $\mathsf{ReallySmall}^{\mathcal{A}} = \mathsf{Child}^{\mathcal{A}} = \mathsf{Ancestor}^{\mathcal{A}} = \varnothing$, that is the only property we know of is that the smurf "schlumpfine" is red, large, and happy. All other predicates are interpreted as the empty set.

   Then $\mathcal{A} \models F$ holds for each of the five sentences above and thus the formalisation is satisfiable.

   The only critical sentences is the first and last one. The first one is satisified as "schlumpfine" is the only smurf and happy. And the last one follows as "schlumpfine" is a red smurf. Formulas ② and ③ are trivially satisfied as the assumptions in the implications are false and formula ④ follows as $\mathsf{ReallySmall}^{\mathcal{A}}$ is empty.

2. a) See Defintion 1.7 in the lecture notes, page 4.

   b) The simplest example of a term t, where the definition is ill-defined would be $t = a$, a a variable.

3. We obtain the following sentences in SNF:

   a)
   - $F_1^S :\Longleftrightarrow \forall y \forall x\ (\neg(x > y) \lor x > \mathsf{f}(x,y)) \land (\neg(x > y) \lor \mathsf{f}(x,y) > y)$.
   - $F_2^S :\Longleftrightarrow \forall x \forall y\ (\neg\mathsf{Q}(y) \lor \mathsf{P}(x))$.
   - $F_3^S :\Longleftrightarrow \forall y\ ((\mathsf{P}(\mathsf{a}) \lor \mathsf{S}(\mathsf{b})) \land (\neg\mathsf{Q}(y) \lor \mathsf{S}(\mathsf{b})) \land (\neg\mathsf{R}(y,\mathsf{a}) \lor \mathsf{S}(\mathsf{b})))$.

   where a, b, f are newly introduced Skolem constants or Skolem functions, respectively.

   b) The claim is incorrect, see Theorem 5.2 on page 37 in the lecture notes for the correct formulation. To see that the claim is wrong consider $F := \exists x\ \mathsf{P}(x)$ and $G := \mathsf{P}(a)$ for some contant a. Observe that the sentence $\exists x\ \mathsf{P}(x) \rightarrow \mathsf{P}(a)$ is not valid.

c) Set $\mathcal{L}' := \mathcal{L} \cup \{\mathsf{d}\}$, where d denotes a new individual constant.

4. a) The formula is unsatisfiable.

   b) In the *ordered* resolution proofs given below the most general unifier employed is written to the right of each applied inference. First we derive the clause $\mathsf{R}(\mathsf{a}, y)$ as follows.

$$\dfrac{\dfrac{\mathsf{P}(x) \lor \mathsf{Q}(x) \lor \mathsf{R}(x, y) \quad \neg \mathsf{P}(x)}{\mathsf{Q}(x) \lor \mathsf{R}(x, y)} \qquad \neg \mathsf{Q}(\mathsf{a})}{\mathsf{R}(\mathsf{a}, y)} \; \sigma = \{x \mapsto \mathsf{a}\}$$

Using this deduction Π, we derive the empty clause.

$$\dfrac{\begin{array}{c}\Pi \\ \mathsf{R}(\mathsf{a}, y)\end{array} \qquad \dfrac{\dfrac{\dfrac{\mathsf{S}(\mathsf{a}, y) \lor \neg \mathsf{R}(\mathsf{a}, y) \lor \mathsf{S}(x, \mathsf{b})}{\mathsf{S}(\mathsf{a}, \mathsf{b}) \lor \neg \mathsf{R}(\mathsf{a}, \mathsf{b})} \, \sigma_1 \qquad \neg \mathsf{S}(\mathsf{a}, \mathsf{b}) \lor \neg \mathsf{R}(\mathsf{a}, \mathsf{b})}{\neg \mathsf{R}(\mathsf{a}, \mathsf{b}) \lor \neg \mathsf{R}(\mathsf{a}, \mathsf{b})}}{\neg \mathsf{R}(\mathsf{a}, \mathsf{b})}}{\Box} \; \sigma_2$$

Here $\sigma_1 := \{x \mapsto \mathsf{a}, y \mapsto \mathsf{b}\}$ and $\sigma_2 := \{y \mapsto \mathsf{b}\}$.

5.

statement	yes	no
Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that the respective universes coincide and suppose $\mathcal{I}_1, \mathcal{I}_2$ coincide on the constants in the closed formula F. Then $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$.	✓	
Let $\mathcal{A}$, $\mathcal{B}$ be structures and $\mathcal{A} \cong \mathcal{B}$. Then for every sentence F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.	✓	
For all formulas F and all sets of formulas $\mathcal{G}$ we have that $\mathcal{G} \models F$ iff $\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$.		✓
Suppose $\mathcal{G}$ is a set of formulas and $\mathcal{G} \models F$. Then there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{G}_0 \models F$.	✓	
Let A, B be sets such that there exists a bijection m between them. Then if $\mathcal{A}$ is a structure with domain A, there exists a structure $\mathcal{B}$ with domain B such that $\mathcal{A} \cong \mathcal{B}$.	✓	
Suppose the sentence $A \to C$ is valid. Then there exists no sentence B such that $A \to B$ and $B \to C$ are valid.		✓
If a set of formulas $\mathcal{G}$ has an infinite model, then $\mathcal{G}$ has no countable infinite model.		✓
If the sentence $A \to C$ holds, then there exists a sentence B such that $A \to B$ and $B \to C$.	✓	
For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $F \approx \forall x_1 \ldots \forall x_n (C_1 \wedge \cdots \wedge C_m)$.	✓	
Reachability in directed graphs is expressible as a second-order formula.	✓	