# Complexity Analysis by Graph Rewriting

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# Outline

# Complexity Analysis by Term Rewriting

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- abstract model of computation rooted in equational theory
- underlies much of declarative programming

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TRS $\mathcal{R}_{\mathsf{D}}$ consists of the rewrite rules

① $\mathsf{D}(c) \to 0$

② $\mathsf{D}(x \times y) \to \mathsf{D}(x) \times y + x \times \mathsf{D}(y)$

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$$\begin{aligned}\mathsf{D}(\,c + (\,c \times c\,)) &\to_{\mathcal{R}_{\mathsf{D}}} \mathsf{D}(\,c\,) + \mathsf{D}(\,c \times c\,) \\ &\to_{\mathcal{R}_{\mathsf{D}}} 0 + \mathsf{D}(c \times c)\end{aligned}$$

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&\to^{2}_{\mathcal{R}_{\mathsf{D}}} 0 + 0 \times c + c \times 0
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## Computation

$$\mathsf{D}(\,\mathsf{c} + (\,\mathsf{c} \times \mathsf{c}\,)) = 0 + 0 \times \mathsf{c} + \mathsf{c} \times 0$$

- $\mathcal{R}_{\mathsf{D}}$ computes differentiation of arithmetical expressions

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## Computation

Let $\mathcal{R}$ denote a confluent and terminating TRS, and let $\mathsf{f}$ be a defined symbol. Then $\mathcal{R}$ computes the function $f$ defined by

$$f(s_1, \ldots, s_n) = t \qquad \Longleftrightarrow \qquad \mathsf{f}(s_1, \ldots, s_n) \to^{!}_{\mathcal{R}} = t$$

for values $s_1, \ldots, s_n$ and $t$. value . . . term build from constructor symbols

# Complexity Analysis by Rewriting

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- derivation length

$$\mathsf{dl}(t, \to) = \max\{\ell \mid \exists(t_1, \ldots, t_\ell).\ t \to t_1 \to \cdots \to t_\ell\}$$

$$\mathsf{dl}(n, T, \to) = \max\{\mathsf{dl}(t, \to) \mid t \in T \text{ and } |t| \leqslant n\}$$

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- runtime complexity

$$\mathsf{rc}_{\mathcal{R}}(n) = \mathsf{dl}(n, \mathcal{B}, \to_{\mathcal{R}})$$

where $\mathcal{B} := \{f(t_1, \ldots, t_n) \mid f \text{ defined and } t_i \text{ values}\}$ denotes the set of basic terms

$$\mathsf{D}(\mathsf{c}) \in \mathcal{B} \qquad \mathsf{D}(\mathsf{c} \times \mathsf{c} + \mathsf{c}) \in \mathcal{B} \qquad \mathsf{D}(\mathsf{D}(\mathsf{D}(\mathsf{c}))) \notin \mathcal{B}$$

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apply rule only if arguments are irreducible

- innermost runtime complexity

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# Automated Complexity Analysis by Rewriting

- direct techniques

  arctic interpretations, polynomial interpretations, matrix interpretations, match- and raise-bounds, polynomial path orders

- transformation techniques

  (weak) dependency pairs (+ argument filtering, dependency graph, reduction pairs, subterm criterion, usable rules, ...), dummy elimination, freezing, mirror, narrowing, reverse, rewriting right-hand sides, (restricted) semantic labeling (root labeling, finite semantic labeling), size change termination, split, strip, type introduction, uncurrying

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- above techniques induce polynomial (innermost) runtime-complexity
- do they assess polynomial computational complexity?

# This Talk

- runtime-complexity is a reasonable cost model for rewriting

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**Theorem**

*Let $\mathcal{R}$ be a confluent and terminating TRS.*
*If $\mathcal{R}$ admits polynomial bounded (innermost) runtime-complexity then the functions computed by $\mathcal{R}$ are polytime-computable.*

$$\mathrm{rc}^{\mathrm{i}}_{\mathcal{R}}(n) \leqslant n^k \Rightarrow f \text{ computed by } \mathcal{R} \text{ is computable in time } \mathrm{O}(n^{5*(k+1)})$$

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- runtime-complexity is a reasonable cost model for rewriting

Example

① $\mathsf{D}(c) \to 0$

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- above TRS $\mathcal{R}_{\mathsf{D}}$ admits polynomial (linear) runtime-complexity
- symbolic differentiation as defined by $\mathcal{R}_{\mathsf{D}}$ is polytime computable

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- a single rewrite step may copy arbitrary large terms

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\begin{aligned}
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\mathsf{D}((c \times c) \times (c \times c)) &= ((0 \times c + c \times 0) \times c + (c \times c) \times 0) \times (c \times c)\\
&\quad + (c \times c) \times ((0 \times c + c \times 0) \times c + (c \times c) \times 0)\\
&\vdots
\end{aligned}
$$

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1. replace copying by sharing

   - encode terms as graphs
   - perform rewriting on graphs

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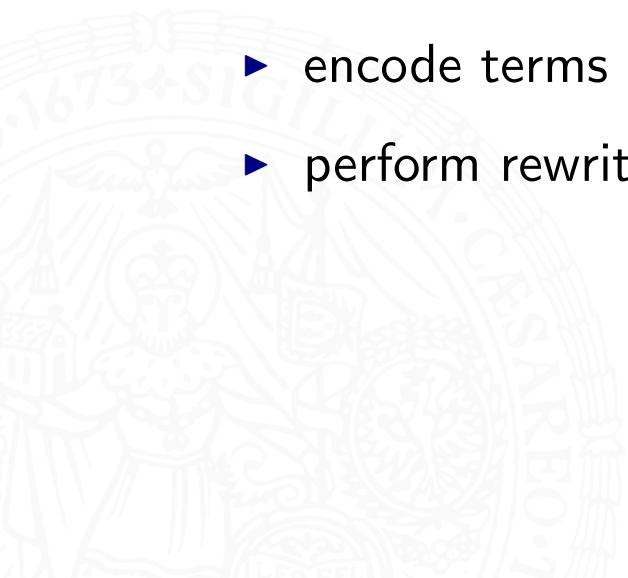

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## Solution

1. replace copying by sharing — graph rewriting
   - encode terms as graphs
   - perform rewriting on graphs
2. implement graph rewriting on a Turing machine

# Graph Rewriting

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## Labeled and Directed Graph

A labeled and directed graph $G = (\mathsf{V}_G, \mathsf{succ}_G, \mathsf{L}_G)$ over labels $\mathcal{L}$ is a structure such that

- $\mathsf{V}_G$ is a finite set of nodes (or vertices)
- $\mathsf{succ}_G : \mathsf{V}_G \to \mathsf{V}_G^*$ maps nodes $n$ to an ordered sequence of nodes, the successors of $n$
- $\mathsf{L}_G : \mathsf{V}_G \to \mathcal{L}$ labels nodes with elements from $\mathcal{L}$

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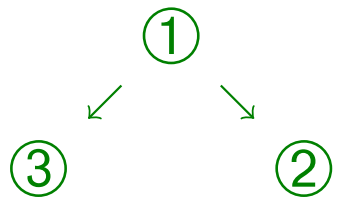
## Example

Let $\mathsf{V} := \{1, 2, 3\}$ and $\mathsf{L} := \{1 \mapsto ①, 2 \mapsto ②, 3 \mapsto ③\}$.


$\mathsf{succ} := \{1 \mapsto [2, 3], 2 \mapsto [], 3 \mapsto []\}$




$\mathsf{succ} := \{1 \mapsto [3, 2], 2 \mapsto [], 3 \mapsto []\}$

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A term graph with respect to a signature $\mathcal{F}$ and variables $\mathcal{V}$ is an acyclic and rooted graph $S = (\mathsf{V}_S, \mathsf{succ}_S, \mathsf{L}_S)$ over labels $\mathcal{F} \cup \mathcal{V}$.

1. if $\mathsf{L}_S(n) = f$ is a $k$-ary function symbol then $\mathsf{succ}_S(n) = [n_1, \ldots, n_k]$
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3. variables are represented by unique nodes

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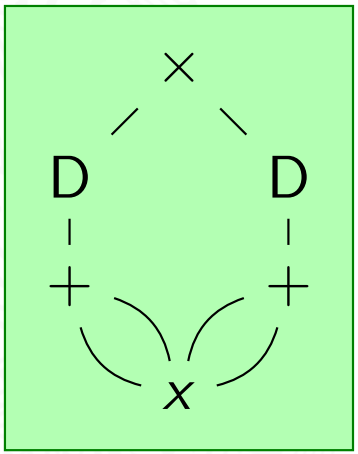


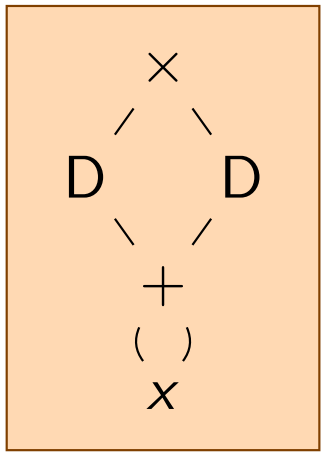


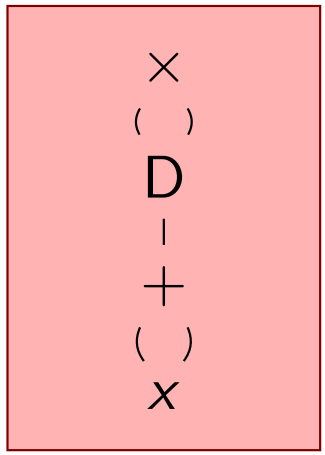

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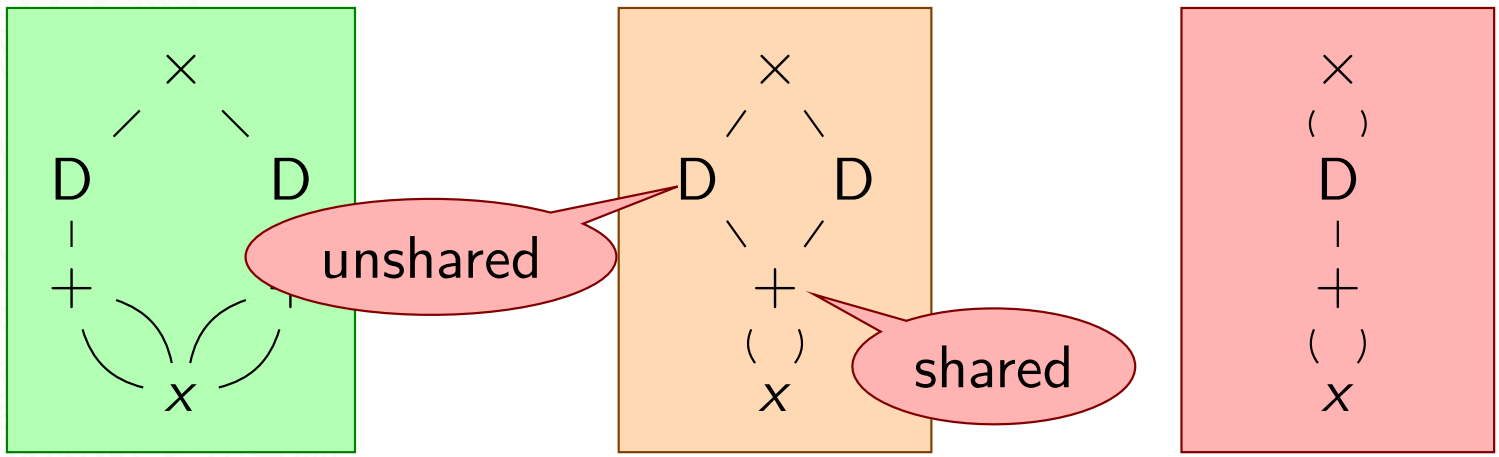

# Graph Rewriting

## Term Graph

A term graph with res … is an acyclic and rooted graph $S =$ …

1. if $\mathsf{L}_S(n) = f$ is a … $[n_1, \ldots, n_k]$
2. if $\mathsf{L}_S(n) = x$ is a …

…que …odes

**minimally sharing**

- only variable nodes are shared

**$(\mathcal{R}_\mathsf{D})$ normal-form sharing**

- nodes representing normal-forms maximally shared
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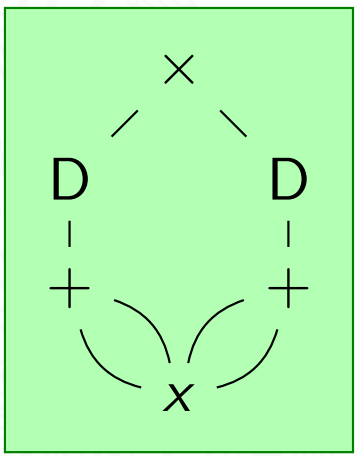


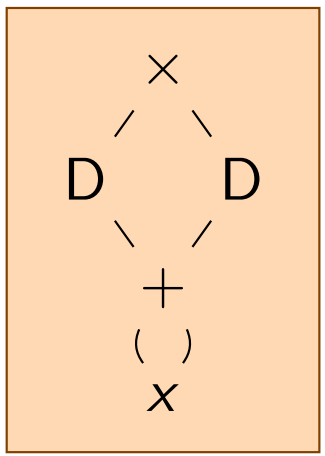


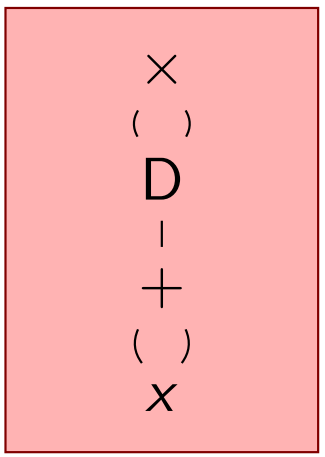

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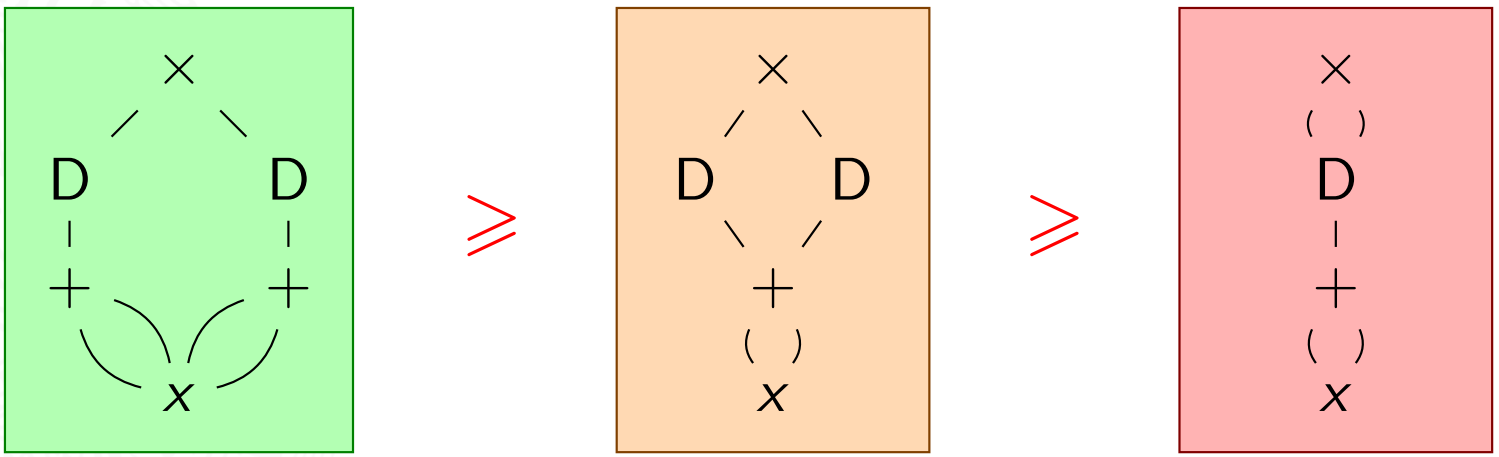

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- a term graph morphism $m\colon S \to T$ is a function $m\colon \mathsf{V}_S \to \mathsf{V}_T$
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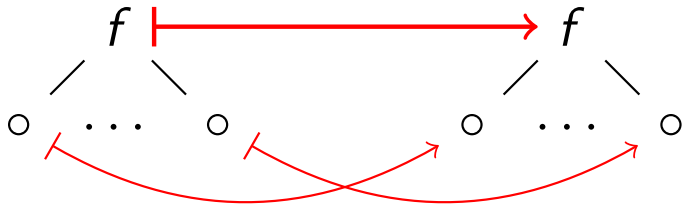

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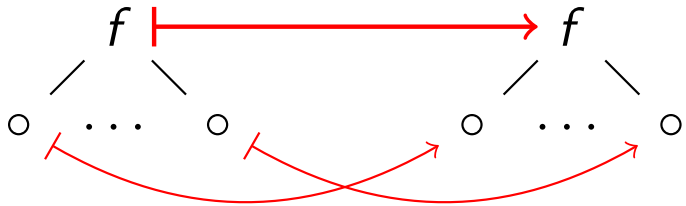

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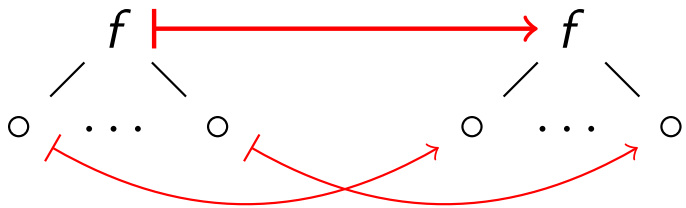


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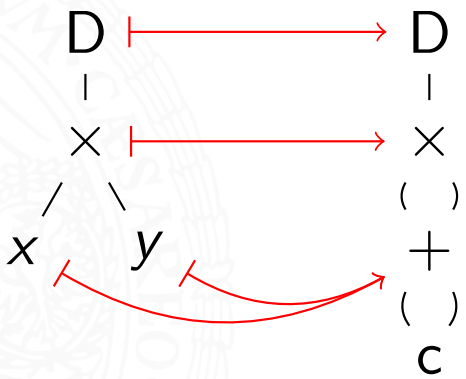

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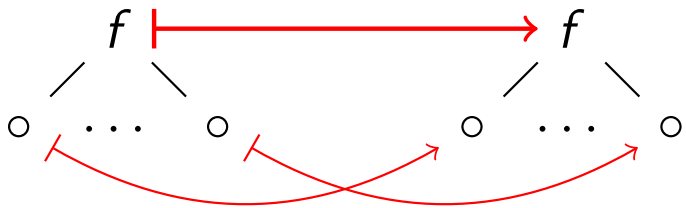


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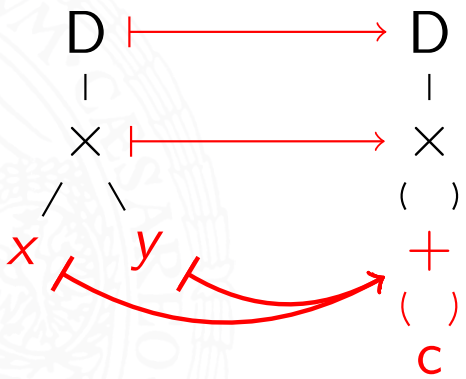

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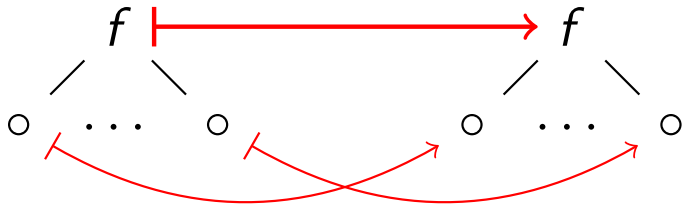


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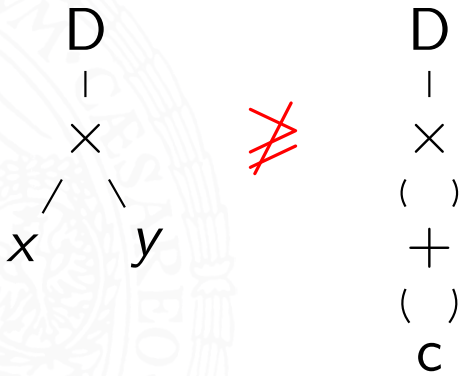

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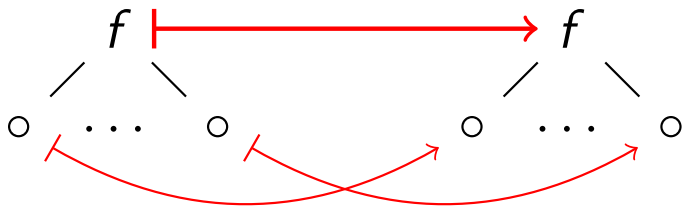


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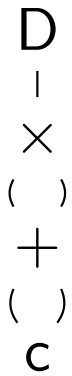


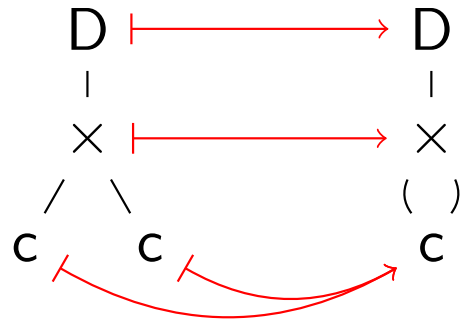

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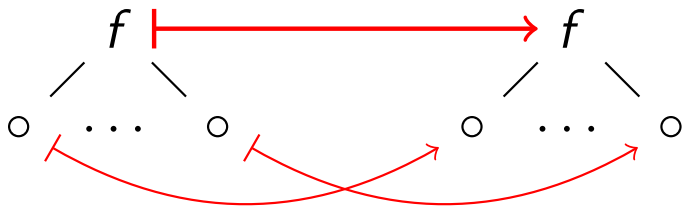


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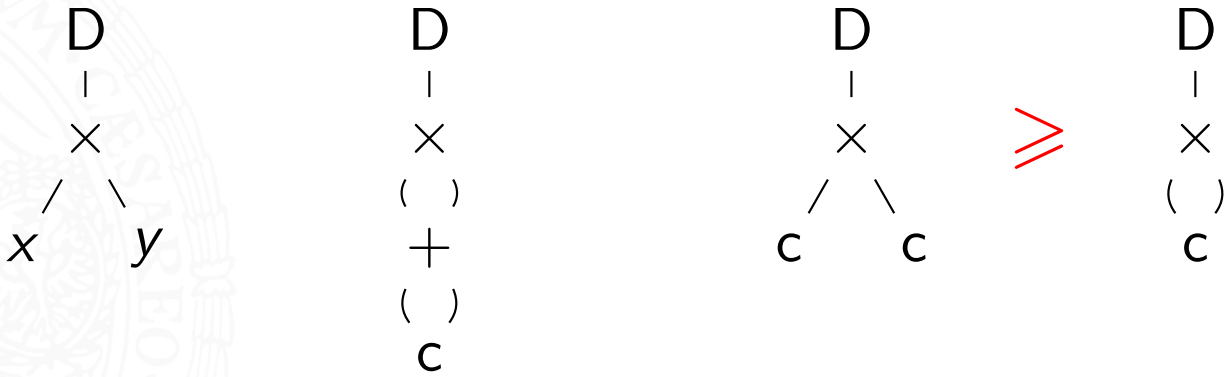

# Graph Rewriting

## Graph Rewrite Rule, Graph Rewrite System, Redex

- a rule is a pair of graphs $L, R$, denoted by $L \longrightarrow R$
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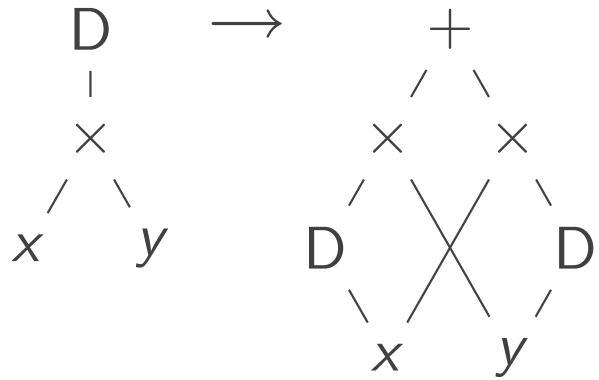

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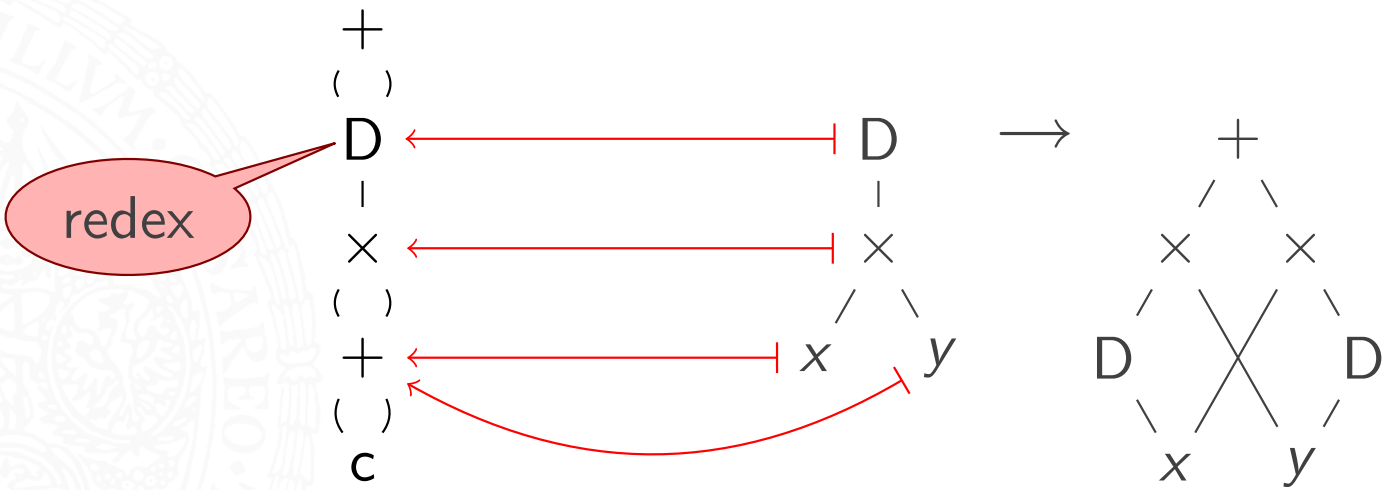

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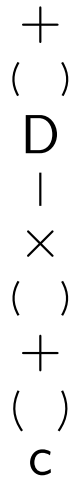


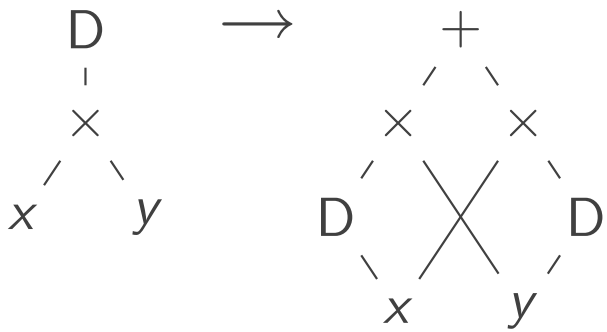

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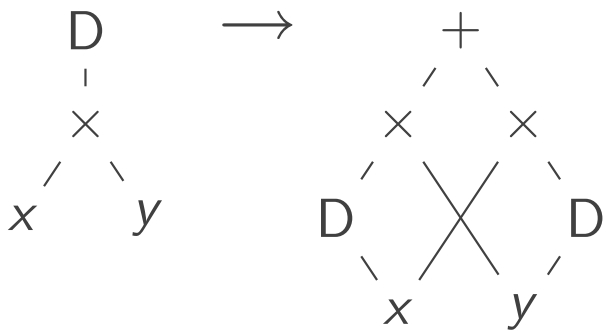

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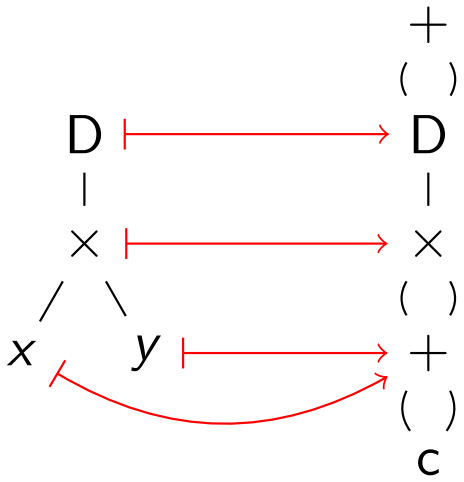


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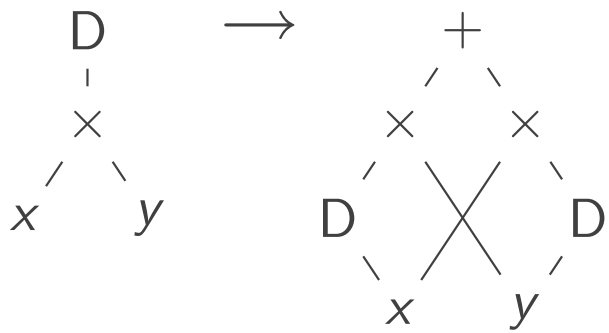

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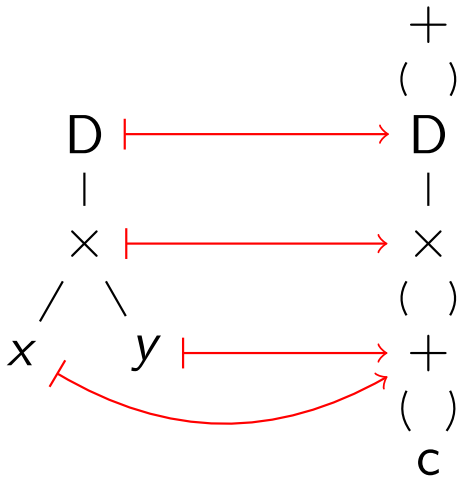


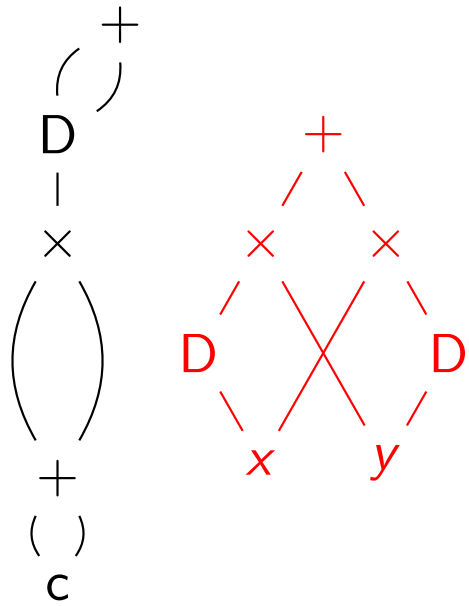


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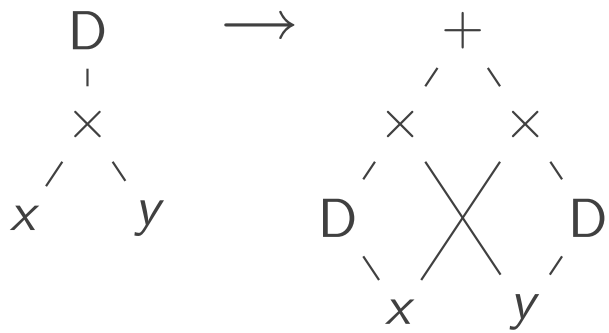

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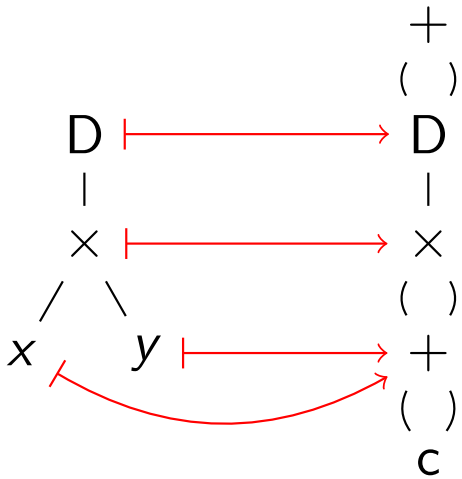


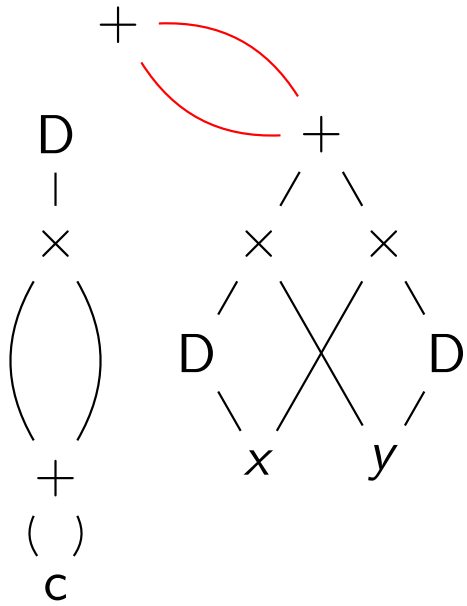


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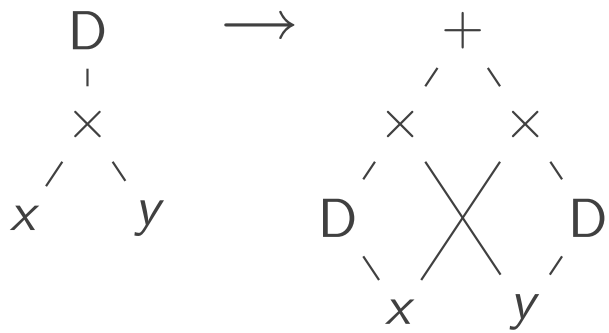

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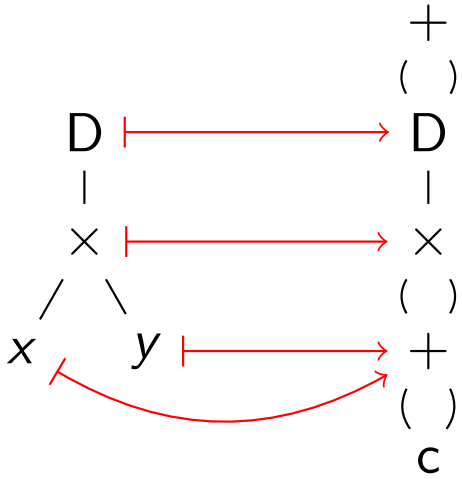


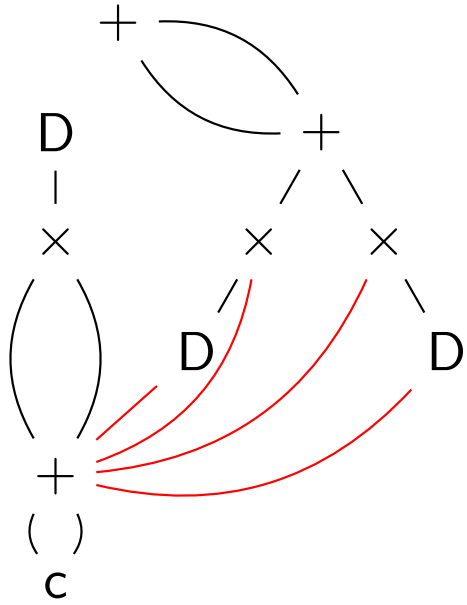


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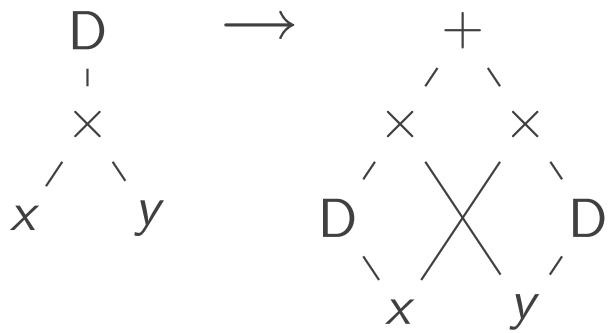

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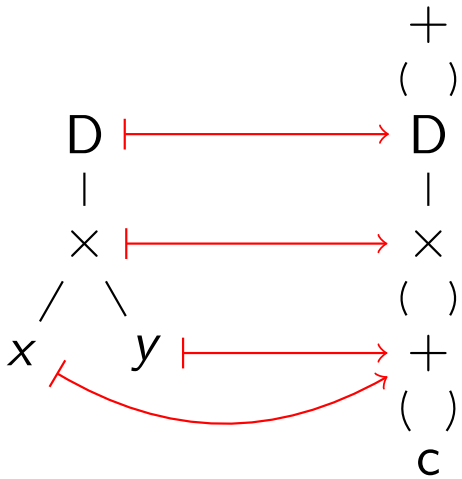


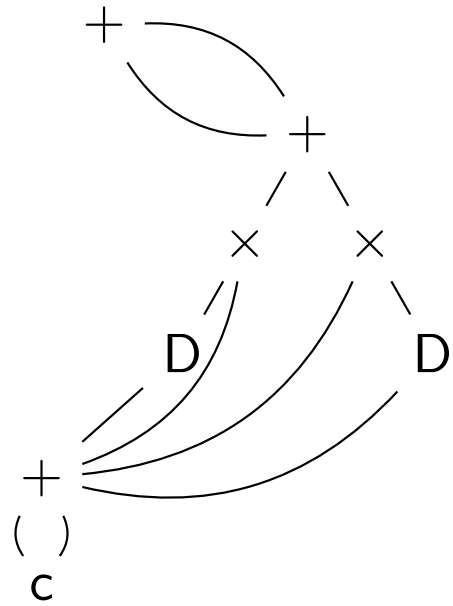


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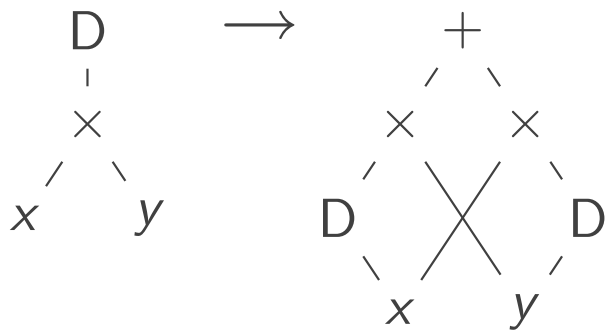

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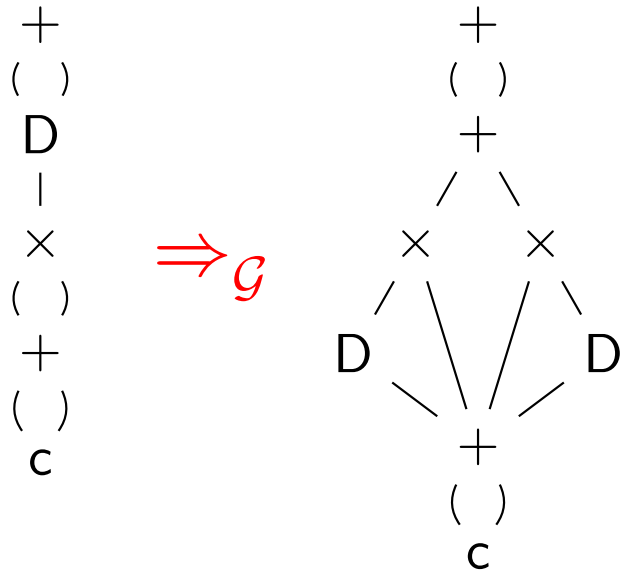


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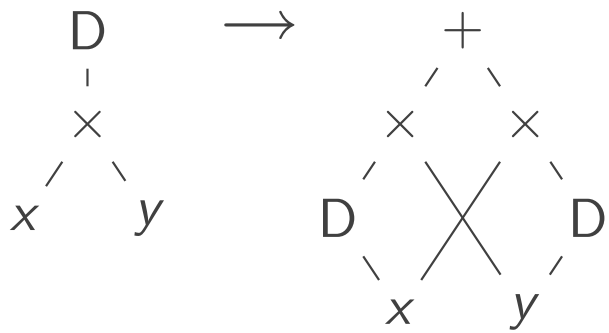

# Adequacy of Graph Rewriting
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Let $\mathcal{R}$ be a TRS. The simulating graph rewrite system $\mathcal{G}(\mathcal{R})$ contains for each rule $l \to r \in \mathcal{R}$ the rule $L \to R$ such that $L$ and $R$ are minimally sharing graphs representing $l$ and $r$ respectively.

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# Adequacy Theorem

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Let $\mathcal{R}$ be a TRS. The simulating graph rewrite system $\mathcal{G}(\mathcal{R})$ contains for each rule $l \to r \in \mathcal{R}$ the rule $L \to R$ such that $L$ and $R$ are minimally sharing graphs representing $l$ and $r$ respectively.

## Problem ②

$$\mathcal{R} := \{\mathsf{a} \to \mathsf{b}\} \qquad \mathcal{G}(\mathcal{R}) := \{\mathsf{a} \to \mathsf{b}\}$$

$$\mathsf{f}(\mathsf{a},\mathsf{a}) \to_{\mathcal{R}} \mathsf{f}(\mathsf{a},\mathsf{b}) \qquad \begin{array}{ccc} \mathsf{f} & & \mathsf{f} \\ (\ \ ) & \Rightarrow_{\mathcal{G}(\mathcal{R})} & (\ \ ) \\ \mathsf{a} & & \mathsf{b} \end{array}$$

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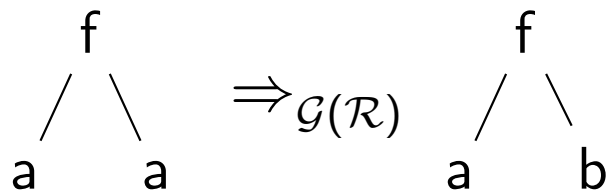

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## Solution

① keep everything below redexes maximally shared

② keep redexes unshared

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*Let $\mathcal{R}$ be a TRS and $\mathcal{G}(\mathcal{R})$ the simulating graph rewrite system of $\mathcal{R}$.*
*Let $s$ be a term and $S$ its normal-form sharing term graph representation.*

$$s \xrightarrow{\mathsf{i}}_{\mathcal{R}} t \qquad \Longleftrightarrow \qquad S \overset{\mathsf{i}}{\Rightarrow}_{\mathcal{G}(\mathcal{R})} \cdot \geqslant T$$

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## Proof Idea

suppose $s = C[l\sigma] \xrightarrow{\mathsf{i}}_{\mathcal{R}} C[r\sigma]$

1. $S \overset{\mathsf{i}}{\Rightarrow}_{\mathcal{G}(\mathcal{R})} T'$ where $T'$ represents $t$

2. $T' \geq T$

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   - thus $m\colon L \to S \restriction p$ for $L \to R \in \mathcal{G}(\mathcal{R})$ representing $l \to r$
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2. $T' \geq T$ $\equiv$ all redex nodes in $T'$ unshared
   - adding $R$ to $S$ does not introduce sharing

□

# Runtime Complexity as Reasonable Cost Model

# Polytime Computability

## Theorem

*Let $\mathcal{R}$ be a confluent and terminating TRS.*
*If $\mathcal{R}$ admits polynomial bounded (innermost) runtime-complexity then the functions computed by $\mathcal{R}$ are polytime-computable.*

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- due to the adequacy theorem
  - $S$ normal-form sharing graph representation of $s = \mathsf{f}(s_1, \ldots, s_n)$
  - $T$ normal-form sharing graph representation of result $t$

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### Difficulty

- efficiency