Solutions

1. Consider the λ-terms $I \stackrel{\text{def}}{=} (\lambda x.\, x)$, $K \stackrel{\text{def}}{=} (\lambda xy.\, x)$, and $\omega \stackrel{\text{def}}{=} (\lambda x.\, x\ x)$.

(12) (a) Reduce the term $\omega\ K\ \omega\ I\ I$ to normal form, using the leftmost innermost reduction strategy.

Solution.

$$\begin{aligned}\omega\ K\ \omega\ I\ I &= \underline{(\lambda x.\, x\ x)\ (\lambda xy.\, x)}\ (\lambda x.\, x\ x)\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda xy.\, x)\ (\lambda xy.\, x)}\ (\lambda x.\, x\ x)\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda yxy.\, x)\ (\lambda x.\, x\ x)}\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda xy.\, x)\ (\lambda x.\, x)}\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda yx.\, x)\ (\lambda x.\, x)}\\ &\to_\beta \lambda x.\, x\end{aligned}$$

(13) (b) Reduce the term $\omega\ K\ \omega\ I\ I$ to normal form, using the leftmost outermost reduction strategy.

Solution.

$$\begin{aligned}\omega\ K\ \omega\ I\ I &= \underline{(\lambda x.\, x\ x)\ (\lambda xy.\, x)}\ (\lambda x.\, x\ x)\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda xy.\, x)\ (\lambda xy.\, x)}\ (\lambda x.\, x\ x)\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda yxy.\, x)\ (\lambda x.\, x\ x)}\ (\lambda x.\, x)\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda xy.\, x)\ (\lambda x.\, x)}\ (\lambda x.\, x)\\ &\to_\beta \underline{(\lambda yx.\, x)\ (\lambda x.\, x)}\\ &\to_\beta \lambda x.\, x\end{aligned}$$

2. Consider the Haskell function:

```
map f []      =  []
map f (x:xs)  =  f x : map f xs
```

Prove by induction over xs that $\texttt{map}\ f\ (\texttt{map}\ g\ xs) = \texttt{map}\ (\lambda x.\, f\ (g\ x))\ xs$ for every list xs.

(5) (a) Give the base case of your induction proof.

Solution.

- **Base Case** ($xs = \texttt{[]}$). $\texttt{map}\ f\ (\texttt{map}\ g\ \texttt{[]}) = \texttt{map}\ f\ \texttt{[]} = \texttt{[]} = \texttt{map}\ (\lambda x.\, f\ (g\ x))\ \texttt{[]}$.

(20) (b) Give the induction hypothesis and the step case of your induction proof.

Solution.

- **Step Case** ($xs = z : zs$). The IH is $\texttt{map}\ f\ (\texttt{map}\ g\ zs) = \texttt{map}\ (\lambda x.\, f\ (g\ x))\ zs$. We conclude by the derivation:

$$\begin{aligned}\texttt{map}\ f\ (\texttt{map}\ g\ (z : zs)) &= \texttt{map}\ f\ (g\ z : \texttt{map}\ g\ zs)\\ &= f\ (g\ z) : \texttt{map}\ f\ (\texttt{map}\ g\ zs)\\ &\stackrel{\text{IH}}{=} f\ (g\ z) : \texttt{map}\ (\lambda x.\, f\ (g\ x))\ zs\\ &= (\lambda x.\, f\ (g\ x))\ z : \texttt{map}\ (\lambda x.\, f\ (g\ x))\ zs\\ &= \texttt{map}\ (\lambda x.\, f\ (g\ x))\ (z : zs)\end{aligned}$$

3. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 2 & \text{if } x = 1, \\ f(x-1) * f(x-2) & \text{otherwise.} \end{cases}$$

(12) (a) Give a direct Haskell implementation of the above function.

Solution.

```
f x | x <= 0      =  1
    | x == 1      =  2
    | otherwise   =  f(x-1) * f(x-2)
```

(13) (b) Use tupling to implement a more efficient variant of f.

Solution.

```
f' x = fst (fpair x)
   where
     fpair x | x <= 0      =  (1, 2)
             | otherwise   =  (z, y * z)
             where
               (y, z) = fpair (x-1)
```

4. Consider the typing environment $E = \{\mathsf{s} :: \mathsf{Int} \to \mathsf{Int}, \mathsf{0} :: \mathsf{Int}\}$.

(10) (a) Transform the type inference problem $E \rhd (\lambda xy.\, x\ (x\ y))\ \mathsf{s}\ \mathsf{0} :: \alpha_0$ into a unification problem.

Solution.

$$E \rhd (\lambda xy.\, x\ (x\ y))\ \mathsf{s}\ \mathsf{0} :: \alpha_0$$
$$\overset{\mathsf{app}}{\Rightarrow}$$
$$E \rhd (\lambda xy.\, x\ (x\ y))\ \mathsf{s} :: \alpha_1 \to \alpha_0; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{app}}{\Rightarrow}$$
$$E \rhd \lambda xy.\, x\ (x\ y) :: \alpha_2 \to \alpha_1 \to \alpha_0; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{abs}}{\Rightarrow}$$
$$E, x :: \alpha_3 \rhd \lambda y.\, x\ (x\ y) :: \alpha_4; \alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{abs}}{\Rightarrow}$$
$$E, x :: \alpha_3, y :: \alpha_5 \rhd x\ (x\ y) :: \alpha_6; \alpha_4 \approx \alpha_5 \to \alpha_6; \alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{app}}{\Rightarrow}$$
$$E, x :: \alpha_3, y :: \alpha_5 \rhd x :: \alpha_7 \to \alpha_6; E, x :: \alpha_3, y :: \alpha_5 \rhd x\ y :: \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{con}}{\Rightarrow}$$

$$\alpha_3 \approx \alpha_7 \to \alpha_6; E, x :: \alpha_3, y :: \alpha_5 \rhd x\ y :: \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{app}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; E, x :: \alpha_3, y :: \alpha_5 \rhd x :: \alpha_8 \to \alpha_7; E, x :: \alpha_3, y :: \alpha_5 \rhd y :: \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; E, x :: \alpha_3, y :: \alpha_5 \rhd y :: \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} :: \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; E \rhd \mathsf{0} :: \alpha_1$$
$$\overset{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$

(10) (b) Solve the following unification problem (if possible).

$$\begin{array}{ll} \alpha_3 \approx \alpha_7 \to \alpha_6 & \alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4 \\ \alpha_3 \approx \alpha_8 \to \alpha_7 & \mathsf{Int} \to \mathsf{Int} \approx \alpha_2 \\ \alpha_5 \approx \alpha_8 & \mathsf{Int} \approx \alpha_1 \\ \alpha_4 \approx \alpha_5 \to \alpha_6 & \end{array}$$

Solution.

$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_3/\alpha_7 \to \alpha_6\}}$$
$$\alpha_7 \to \alpha_6 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_7 \to \alpha_6) \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$

$$\alpha_7 \approx \alpha_8; \alpha_6 \approx \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_7 \to \alpha_6) \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_7/\alpha_8\}}$$
$$\alpha_6 \approx \alpha_8; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_6) \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_6/\alpha_8\}}$$
$$\alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_8;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_5/\alpha_8\}}$$
$$\alpha_4 \approx \alpha_8 \to \alpha_8;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_4; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_4/\alpha_8 \to \alpha_8\}}$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_8 \to \alpha_8; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$
$$\alpha_2 \approx \alpha_8 \to \alpha_8; \alpha_1 \to \alpha_0 \approx \alpha_8 \to \alpha_8; \mathsf{Int} \to \mathsf{Int} \approx \alpha_2; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_2/\alpha_8 \to \alpha_8\}}$$
$$\alpha_1 \to \alpha_0 \approx \alpha_8 \to \alpha_8; \mathsf{Int} \to \mathsf{Int} \approx \alpha_8 \to \alpha_8; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$
$$\alpha_1 \approx \alpha_8; \alpha_0 \approx \alpha_8; \mathsf{Int} \to \mathsf{Int} \approx \alpha_8 \to \alpha_8; \mathsf{Int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_1/\alpha_8\}}$$
$$\alpha_0 \approx \alpha_8; \mathsf{Int} \to \mathsf{Int} \approx \alpha_8 \to \alpha_8; \mathsf{Int} \approx \alpha_8$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_0/\alpha_8\}}$$
$$\mathsf{Int} \to \mathsf{Int} \approx \alpha_8 \to \alpha_8; \mathsf{Int} \approx \alpha_8$$
$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$
$$\mathsf{Int} \approx \alpha_8; \mathsf{Int} \approx \alpha_8; \mathsf{Int} \approx \alpha_8$$
$$\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_8/\mathsf{Int}\}}$$
$$\mathsf{Int} \approx \mathsf{Int}; \mathsf{Int} \approx \mathsf{Int}$$
$$\Rightarrow^{(\mathsf{t})}_{\iota}$$
$$\mathsf{Int} \approx \mathsf{Int}$$
$$\Rightarrow^{(\mathsf{t})}_{\iota}$$
$$\square$$

This leads to the solution

$$\{\alpha_0/\mathsf{Int}, \alpha_1/\mathsf{Int}, \alpha_2/\mathsf{Int} \to \mathsf{Int}, \alpha_3/\mathsf{Int} \to \mathsf{Int}, \alpha_4/\mathsf{Int} \to \mathsf{Int}, \alpha_5/\mathsf{Int}, \alpha_6/\mathsf{Int}, \alpha_7/\mathsf{Int}, \alpha_8/\mathsf{Int}\}$$