Functional Programming

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Today's Topics

- Evaluation Strategies
- Abstract Data Types
- Sets and Binary Search Trees

Evaluation Strategies

Recall - λ-Terms

$$t \stackrel{\text{def}}{=} x \mid (\lambda x.\, t) \mid (t\ t)$$

Examples

Conventions	Verbose	in Words
$x\ y$	$(x\ y)$	"x applied to y"
$\lambda x.\, x$	$(\lambda x.\, x)$	"lambda x to x" (identity function)
$\lambda xy.\, x$	$(\lambda x.\, (\lambda y.\, x))$	"lambda x y to x"
$\lambda x.\, x\ x$	$(\lambda x.\, (x\ x))$	"lambda x to x applied to x"
$(\lambda x.\, x)\ x$	$((\lambda x.\, x)\ x)$	"lambda x to x, applied to x"

Recall - β-Reduction

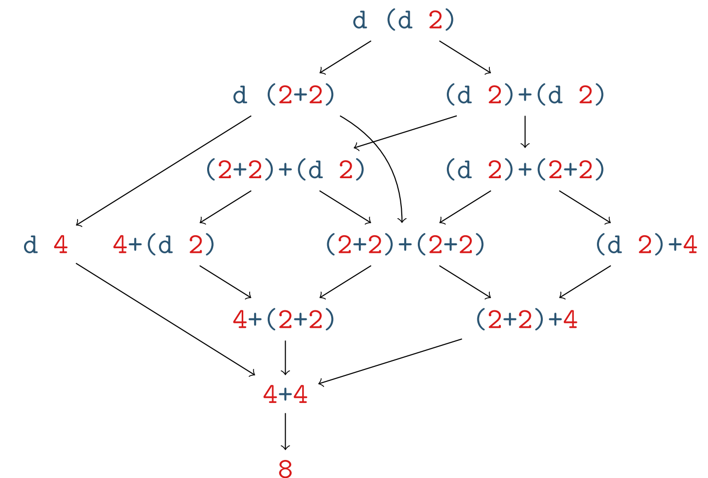
- term s (β-)reduces to term t in one step
- written: $s \rightarrow_\beta t$
- iff there is context C, variable x, and terms u and v, s.t.,
- $s = C[(\lambda x.\, u)\ v]$ and $t = C[u\{x/v\}]$

Examples

$$K \stackrel{\text{def}}{=} \lambda xy.\, x$$
$$I \stackrel{\text{def}}{=} \lambda x.\, x$$
$$\Omega \stackrel{\text{def}}{=} (\lambda x.\, x\ x)\ (\lambda x.\, x\ x)$$

Order of Evaluation

- consider `d x = x + x`
- the term `d (d 2)` may be evaluated as follows



Strategies

- fix evaluation order
- call-by-value (compute arguments before function calls)
- call-by-name (compute arguments "on demand")

Example

- call-by-value

`d (d 2)` = `d (2+2)`
= `d 4`
= `4+4`
= `8`

- call-by-name

`d (d 2)` = `(d 2) + (d 2)`
= `(2+2) + (d 2)`
= `4 + (d 2)`
= `4 + (2+2)`
= `4 + 4`
= `8`

(Leftmost) Innermost Reduction

- always reduce leftmost innermost redex
- a redex u inside a term t is innermost if it does not contain any redexes as proper subterms, i.e.,

$$\nexists C\ s.\, u = C[s],\ C \neq \square \text{ and } s \text{ is a redex}$$

Example

- consider $t = (\lambda x.\, (\lambda y.\, y)\ x)\ z$
- $(\lambda y.\, y)\ x$ is innermost redex
- t is redex, but not an innermost redex

(Leftmost) Outermost Reduction

- always reduce leftmost outermost redex
- a redex u inside a term t is outermost if it is not a proper subterm of some other redex inside t, i.e.,

$$\nexists D\ C\ s.\ t = D[s],\ s = C[u],\ C \neq \Box \text{ and } s \text{ is a redex}$$

Example

- consider $t = (\lambda x.\,(\lambda y.\,y)\ x)\ z$
- t is an outermost redex
- $(\lambda y.\,y)\ x$ is redex, but not outermost redex

Exercises

- consider the λ-terms
- $S = \lambda xyz.\,x\ z\ (y\ z)$
- $K = \lambda xy.\,x$
- $I = \lambda x.\,x$
- reduce $S\ K\ I$ to NF using leftmost innermost reduction
- reduce $S\ K\ I$ to NF using leftmost outermost reduction

Call-by-Value

- use innermost reduction
- corresponds to strict (or eager) evaluation
- adopted by most programming languages
- slight modification: only reduce terms not in WHNF

Call-by-Name

- use outermost reduction
- corresponds to lazy evaluation (without memoization)
- e.g., adopted by Haskell
- slight modification: again, only reduce terms not in WHNF

Weak Head Normal Form

term t is in weak head normal form iff t is not an application

Abstract Data Types

Idea

- hide implementation details
- just provide interface
- allows to change implementation (e.g., make more efficient) without breaking client code

Haskell

- consider module
```
module M (T, ...) where
type T = C1 |   | CN
```
- only name `T` is exported, but none of `C1` to `CN`
- thus we are not able to directly construct values of type `T`
- if we want to export `C1` to `CN`, we can use `T(..)` in export list

Set Characteristics

- order of elements not important
- no duplicates

Examples

$$\{1,2,3,5\} = \{5,1,3,2\}$$
$$\{1,1,2,2\} = \{1,2\}$$

Set Operations

description	notation	Haskell
empty set	$\varnothing$	`empty :: Set a`
insertion	$\{x\} \cup S$	`insert :: a -> Set a -> Set a`
membership	$e \in S$	`mem :: a -> Set a -> Bool`
union	$S \cup T$	`union :: Set a -> Set a -> Set a`
difference	$S \setminus T$	`diff :: Set a -> Set a -> Set a`

Example - Sets as Lists

```
module Set (Set,empty,insert,mem,union,diff) where
import qualified Data.List as List
data Set a = Set [a]

empty :: Set a
empty = Set []

insert :: Eq a => a -> Set a -> Set a
insert x (Set xs) = Set (List.nub (x : xs))

mem :: Eq a => a -> Set a -> Bool
mem x (Set xs) = x `elem` xs

union :: Eq a => Set a -> Set a -> Set a
union (Set xs) (Set ys) = Set (List.nub (xs ++ ys))

diff :: Eq a => Set a -> Set a -> Set a
diff (Set xs) (Set ys) = Set (xs List.\\ ys)
```


Note - Imports

- `import M` imports all functions and types defined in module `M`
- we may restrict to `f1`, ..., `fN`, writing
`import M (f1,...,fN)`
- by `import M hiding (f1,...,fN)` we import everything except the functions `f1` to `fN`
- `import qualified M` allows to access all functions defined in `M` using prefix `M.`
- `import qualified M as N`, same as `import qualified M` but additionally rename `M` to `N`

New Types

- in `Set` we use **data** with a single constructor `Set` to hide the fact that sets are implemented by lists
- this is a common special case
- we may use `newtype Set a = Set a` instead
- only difference: **newtype** has better performance than **data**

Record Syntax

- for data type / new type `T`, instead of `C t1 ...tN`, we may use
- `C {n1 :: t1, ..., nN :: tN}` as constructor
- provides selector functions `n1::T -> t1`, ..., `nN::T -> tN`

Example

- `data Equation a = E { lhs :: a, rhs :: a }`

```
ghci> let e1 = E "10" "5+5"
ghci> let e2 = E { rhs = "5+5", lhs = "10" }
```

-
```
ghci> lhs e1
"10"
ghci> rhs e2
"5+5"
```


Sets and Binary Search Trees

The Type

- we want to use type `BTree` without prefix

```
import BTree (BTree)
```

- all other functions from `BTree` with prefix

```
import qualified BTree
```

- the internal representation of a set is a binary tree

```
newtype Set a = Set { rep :: BTree a }
```

Note

- `newtype Set a = Set { rep :: BTree a }` is almost the same as writing `type Set a = BTree a`
- additionally the type system prevents us from "accidentally" (i.e., without the constructor `Set`) using `BTree`s as `Set`s
- no runtime penalty (in contrast to `data Set a = Set { rep :: BTree }`)
- reason: **newtype** restricted to single constructor (usually of same name as newly introduced type),
- whereas **data** may define arbitrary many constructors (e.g., `Empty` and `Node`)

Empty Set

```
empty :: Set a
empty = Set BTree.Empty
```

Membership

```
mem :: Ord a => a -> Set a -> Bool
mem x s = x `memTree` (rep s)
  where memTree x Empty = False
        memTree x (Node y l r) =
          case compare x y of
            EQ -> True
            LT -> x `memTree` l
            GT -> x `memTree` r
```

Insertion

```
insert :: Ord a => a -> Set a -> Set a
insert x s = Set (insertTree x (rep s))

insertTree :: Ord a => a -> BTree a -> BTree a
insertTree x Empty        = Node x Empty Empty
insertTree x (Node y l r) =
  case compare x y of
    EQ -> Node y l r
    LT -> Node y (insertTree x l) r
    GT -> Node y l (insertTree x r)
```


Union

```
union :: Ord a => Set a -> Set a -> Set a
union s t = Set (rep s `unionTree` rep t)

unionTree :: Ord a => BTree a -> BTree a -> BTree a
unionTree Empty s        = s
unionTree (Node x l r) s =
  insertTree x (l `unionTree` r `unionTree` s)
```


Removing the Maximal Element

```
splitMaxTree :: BTree a -> Maybe (a,BTree a)
splitMaxTree Empty            = Nothing
splitMaxTree (Node x l Empty) = Just (x,l)
splitMaxTree (Node x l r)     =
  let Just (m,r') = splitMaxTree r
  in Just (m,Node x l r')
```

The Maybe Type

- Prelude: `data Maybe a = Just a | Nothing`
- used for type-safe error handling
- if an error occurs, we return `Nothing`
- otherwise `Just` the result

Example - Safe Head

```
safeHead (x:_) = Just x
safeHead _     = Nothing
```


Remove Given Element

```
removeTree :: Ord a => a -> BTree a -> BTree a
removeTree x Empty        = Empty
removeTree x (Node y l r) = case compare x y of
  LT -> Node y (removeTree x l) r
  GT -> Node y l (removeTree x r)
  EQ -> case splitMaxTree l of
    Nothing     -> r
    Just (m,l') -> Node m l' r
```

Idea

- have binary search tree (BST)
- `x` smaller `y`: `x` can only occur in `l`
- `x` greater `y`: `x` can only occur in `r`
- `x` equals `y`: remove current node and
- combine `l` and `r` into new BST
- therefore, take maximum of `l` as new root
- guarantees that all other elements in `l` are smaller and
- that all elements in `r` are greater

Difference

```
diff :: Ord a => Set a -> Set a -> Set a
diff s t = Set (rep s `diffTree` rep t)

diffTree :: Ord a => BTree a -> BTree a -> BTree a
diffTree t Empty        = t
diffTree t (Node x l r) =
  removeTree x t `diffTree` l `diffTree` r
```


Exercises (for November 19th)

1. Read chapter 3 of Real World Haskell and the lecture notes about the lambda-calculus.
2. Reduce each of the following λ-terms to NF

$$(\lambda w.\, w)\ ((\lambda xy.\, y)\ (z\ z))$$
$$(\lambda xy.\, x)\ (\lambda z.\, y\ z)$$
$$\lambda z.\, (\lambda x.\, x\ z\ y)\ (\lambda xy.\, y\ z)$$
$$\lambda xy.\, y\ (\lambda w.\, w)\ (\lambda yz.\, y\ x)$$

3. Reduce ADD 3 2 to WHNF using leftmost innermost/outermost reduction.
4. Give λ-terms encoding `(&&)`, `(||)`, and `not`.
5. Implement safe versions (i.e., using `Maybe`) of `tail`, `init`, and `last`.
6. Implement the function `equals :: Ord a => Set a -> Set a -> Bool`, checking whether two sets are equal.