Functional Programming
Selected Solutions Week 6

(for November 19, 2010)

2.

$$
\begin{aligned}
&(\lambda w.\, w)\ ((\lambda x\ y.\, y)\ (z\ z)) \\
&= (\lambda w.\, w)\ (\underline{(\lambda x\ y.\, y)\ (z\ z)}) \\
&\to_\beta \underline{(\lambda w.\, w)\ (\lambda y.\, y)} \\
&\to_\beta \lambda y.\, y
\end{aligned}
$$

$$
\begin{aligned}
&(\lambda x\ y.\, x)\ (\lambda z.\, y\ z) \\
&= \underline{(\lambda x\ y.\, x)\ (\lambda z.\, y\ z)} \\
&\to_\beta \lambda y_1\ z.\, y\ z
\end{aligned}
$$

$$
\begin{aligned}
&\lambda z.\, (\lambda x.\, x\ z\ y)\ (\lambda xy.\, y\ z) \\
&= \lambda z.\, \underline{(\lambda x.\, x\ z\ y)\ (\lambda xy.\, y\ z)} \\
&\to_\beta \lambda z.\, \underline{(\lambda xy.\, y\ z)\ z}\ y \\
&\to_\beta \lambda z.\, \underline{(\lambda y.\, y\ z)\ y} \\
&\to_\beta \lambda z.\, y\ z
\end{aligned}
$$

$$
\begin{aligned}
&\lambda xy.\, y\ (\lambda w.\, w)\ (\lambda yz.\, y\ x) \\
&\to_\beta \lambda xy.\, y\ (\lambda w.\, w)\ (\lambda y\ z.\, y\ x)
\end{aligned}
$$

3. For leftmost innermost reduction we obtain:

$$
\begin{aligned}
&\mathsf{ADD}\ 3\ 2 \\
&= \underline{(\lambda mnfx.\, m\ f\ (n\ f\ x))\ (\lambda fx.\, f\ (f\ (f\ x)))}\ (\lambda fx.\, f\ (f\ x)) \\
&\to_\beta (\lambda nfx.\, \underline{(\lambda fx.\, f\ (f\ (f\ x)))\ f}\ (n\ f\ x))\ (\lambda fx.\, f\ (f\ x)) \\
&\to_\beta (\lambda nfx.\, \underline{(\lambda x.\, f\ (f\ (f\ x)))\ (n\ f\ x)})\ (\lambda fx.\, f\ (f\ x)) \\
&\to_\beta \underline{(\lambda nfx.\, f\ (f\ (f\ (n\ f\ x))))\ (\lambda fx.\, f\ (f\ x))} \\
&\to_\beta \lambda fx.\, f\ (f\ (f\ (\underline{(\lambda fx.\, f\ (f\ x))\ f}\ x))) \\
&\to_\beta \lambda fx.\, f\ (f\ (f\ (\underline{(\lambda x.\, f\ (f\ x))\ x}))) \\
&\to_\beta \lambda fx.\, f\ (f\ (f\ (f\ (f\ x))))
\end{aligned}
$$

And for leftmost outermost:

$$
\begin{aligned}
&\mathsf{ADD}\ 3\ 2\\
&= \underline{(\lambda mnfx.\, m\ f\ (n\ f\ x))\ (\lambda fx.\, f\ (f\ (f\ x)))}\ (\lambda fx.\, f\ (f\ x))\\
&\rightarrow_\beta \underline{(\lambda nfx.\, (\lambda fx.\, f\ (f\ (f\ x)))\ f\ (n\ f\ x))\ (\lambda fx.\, f\ (f\ x))}\\
&\rightarrow_\beta \lambda fx.\, \underline{(\lambda fx.\, f\ (f\ (f\ x)))\ f}\ ((\lambda fx.\, f\ (f\ x))\ f\ x)\\
&\rightarrow_\beta \lambda fx.\, \underline{(\lambda x.\, f\ (f\ (f\ x)))\ ((\lambda fx.\, f\ (f\ x))\ f\ x)}\\
&\rightarrow_\beta \lambda fx.\, f\ (f\ (f\ (\underline{(\lambda fx.\, f\ (f\ x))\ f}\ x)))\\
&\rightarrow_\beta \lambda fx.\, f\ (f\ (f\ (\underline{(\lambda x.\, f\ (f\ x))\ x})))\\
&\rightarrow_\beta \lambda fx.\, f\ (f\ (f\ (f\ (f\ x))))
\end{aligned}
$$

4. Using the definitions $\mathsf{TRUE} \stackrel{\text{def}}{=} \lambda xy.\, x$, $\mathsf{FALSE} \stackrel{\text{def}}{=} \lambda xy.\, y$, and $\mathsf{IF} \stackrel{\text{def}}{=} \lambda xyz.\, x\ y\ z$, we can define the logical connectives as follows:

$$
\begin{aligned}
\mathsf{AND} &\stackrel{\text{def}}{=} \lambda xy.\, \mathsf{IF}\ x\ y\ \mathsf{FALSE} && \rightarrow_\beta \lambda xy.\, x\ y\ \mathsf{FALSE}\\
\mathsf{OR} &\stackrel{\text{def}}{=} \lambda xy.\, \mathsf{IF}\ x\ \mathsf{TRUE}\ y && \rightarrow_\beta \lambda xy.\, x\ \mathsf{TRUE}\ y\\
\mathsf{NOT} &\stackrel{\text{def}}{=} \lambda x.\, \mathsf{IF}\ x\ \mathsf{FALSE}\ \mathsf{TRUE} && \rightarrow_\beta \lambda x.\, x\ \mathsf{FALSE}\ \mathsf{TRUE}
\end{aligned}
$$