Functional Programming
Selected Solutions Week 8
(for December 10, 2010)

4. Recall the definitions of `range` and `rng`:

```
range m n | m > n     = []
          | otherwise = m : range (m+1) n

rng xs m n | m > n     = xs
           | otherwise = rng (n:xs) m (n-1)
```

We first prove an auxiliary lemma about `range`, namely

$$m \leq n \implies \texttt{range}\ m\ (n-1)\ \texttt{++}\ [n] = \texttt{range}\ m\ n \qquad (\star)$$

Here, we use mathematical induction over $|n - m|$ (that is, the absolute value of the difference $n - m$; this guarantees that we cannot slip below 0):

- **Base Case** ($|n - m| = 0$). Since $m \leq n$, we have $m = n$, and thus

$$\begin{aligned} \texttt{range}\ m\ (n-1)\ \texttt{++}\ [n] &= \texttt{[]}\ \texttt{++}\ [n] && (\text{since } m > m - 1) \\ &= [n] \\ &= m\ \texttt{:}\ \texttt{[]} \\ &= m\ \texttt{:}\ \texttt{range}\ (m+1)\ n && (\text{since } m + 1 > m) \\ &= \texttt{range}\ m\ n && (\text{since } m \not> m) \end{aligned}$$

- **Step Case** ($|n - m| = k + 1$). We obtain the IH

$$\texttt{range}\ (m+1)\ (n-1)\ \texttt{++}\ [n] = \texttt{range}\ (m+1)\ n$$

since $|n - (m + 1)| = k$. The proof concludes as follows:

$$\begin{aligned} \texttt{range}\ m\ (n-1)\ \texttt{++}\ [n] &= m\ \texttt{:}\ \texttt{range}\ (m+1)\ (n-1)\ \texttt{++}\ [n] \\ &\overset{\textsf{IH}}{=} m\ \texttt{:}\ \texttt{range}\ (m+1)\ n \\ &= \texttt{range}\ m\ n \end{aligned}$$

Having the auxiliary lemma, we now prove the following equivalence (a trivial consequence of which is `range'` $m\ n =$ `range` $m\ n$):

$$\texttt{rng}\ xs\ m\ n = \texttt{range}\ m\ n\ \texttt{++}\ xs$$

Again, we use mathematical induction over $|n - m|$.

- **Base Case** ($|n-m|=0$). Hence, $m=n$, and thus

$$\begin{aligned}\texttt{rng}\ xs\ m\ n &= \texttt{rng}\ (n\ \texttt{:}\ xs)\ m\ (n-1)\\ &= n\ \texttt{:}\ xs\\ &= n\ \texttt{:}\ \texttt{[]}\ \texttt{++}\ xs\\ &= m\ \texttt{:}\ \texttt{range}\ (m+1)\ n\ \texttt{++}\ xs\\ &= \texttt{range}\ m\ n\ \texttt{++}\ xs\end{aligned}$$

- **Step Case** ($|n-m|=k+1$). We obtain the IH

$$\texttt{rng}\ (n\ \texttt{:}\ xs)\ m\ (n-1) = \texttt{range}\ m\ (n-1)\ \texttt{++}\ (n\ \texttt{:}\ xs)$$

since $|(n-1)-m|=k$. The proof concludes by a case distinction. If $m>n$, we have

$$\begin{aligned}\texttt{rng}\ xs\ m\ n &= xs\\ &= \texttt{[]}\ \texttt{++}\ xs\\ &= \texttt{range}\ m\ n\ \texttt{++}\ xs\end{aligned}$$

Otherwise, we have $m \leq n$ (which lets us apply $\star$).

$$\begin{aligned}\texttt{rng}\ xs\ m\ n &= \texttt{rng}\ (n\ \texttt{:}\ xs)\ m\ (n-1)\\ &\overset{\text{IH}}{=} \texttt{range}\ m\ (n-1)\ \texttt{++}\ (n\ \texttt{:}\ xs)\\ &= (\texttt{range}\ m\ (n-1)\ \texttt{++}\ [n])\ \texttt{++}\ xs\\ &= \texttt{range}\ m\ n\ \texttt{++}\ xs && \text{(by } \star\text{)}\end{aligned}$$

6. We use induction over the structure of xs.

- **Base Case** ($xs = \texttt{[]}$). Trivial.
- **Step Case** ($xs = y\ \texttt{:}\ ys$). If $n=0$, the equivalence does trivially hold. Otherwise, we have $n=k+1$ for some k and the IH

$$\texttt{splitAt'}\ k\ ys = (\texttt{take}\ k\ ys, \texttt{drop}\ k\ ys)$$

The proof concludes as follows:

$$\begin{aligned}\texttt{splitAt'}\ n\ xs &= (y\ \texttt{:}\ zs, ws) && \text{(for } (zs, ws) = \texttt{splitAt'}\ k\ ys\text{)}\\ &\overset{\text{IH}}{=} (y\ \texttt{:}\ \texttt{take}\ k\ ys, \texttt{drop}\ k\ ys)\\ &= (\texttt{take}\ n\ xs, \texttt{drop}\ n\ xs)\end{aligned}$$