Computation of Market Equilibria by Convex Programming

Agenda

- Model Definition
- Solving the Fisher Model
- Solving Exchange Economies
- Potential Application

Motivation

- Markets dominate our economic world
  - In many cases stable price are established
  - win-win situation for (all) participants

- Today’s goal: present models for markets
  - Define stable conditions (equilibria)
  - Define algorithms to compute those
  - Ultimately: use models to cope with similar problems

4

Exchange Economies

The Model (1)

- A market M is given by
  - n goods and
  - m economic agents / traders
    - each trader i has a concave utility function

$$u_i : \mathbb{R}_+^n \to \mathbb{R}_+$$

    - and an initial endowment

$$w_i = (w_{i1}, \ldots, w_{in}) \in \mathbb{R}_+^n$$

The Model (2)

- At given prizes

$$\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n_+$$

- trader i will sell her endowment w_i, and get the bundle of goods

$$x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n_+$$

- maximizing $u_i(x)$ subject to the *budget constrain*

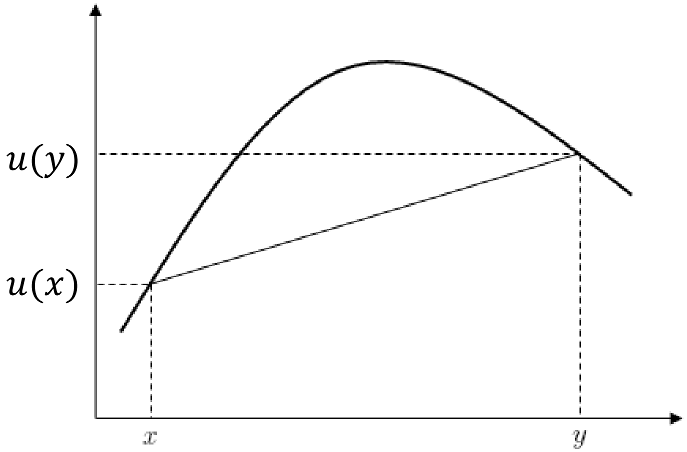
$$\pi \cdot x_i \leq \pi \cdot w_i$$

Market Equilibrium

- An equilibrium is a vector of prices
  $$\pi = (\pi_1, \dots, \pi_n) \in \mathbb{R}_+^n$$
  such that:
  - for each trader i, there is a bundle $\bar{x}_i = (\bar{x}_i, \dots, \bar{x}_i)$ of goods
  - the vector $\bar{x}_i$ is maximizing $u_i\ (x)$ subject to the constrains $\pi \cdot x_i \leq \pi \cdot w_i$ and $\mathrm{x_i} \in \mathbb{R}_+^n$
  - for each good j, $\sum_i \bar{x}_{ij} \leq \sum_i w_{ij}$
- Arrow and Debreu('54): such an equilibrium exists
  - Problem: how to computing such equilibria?

Utility Functions (1)

- Basic assumptions
  - concave:



  - non-satiable: $\forall x \in \mathbb{R}_+^n. \exists y \in \mathbb{R}_+^n : u(y) > u(x)$
  - monoton: $y \geq x \Rightarrow u(y) \geq u(x)$

- Frequent assumption
  - homogeneous $\forall x, \alpha > 0 : u(\alpha x) = \alpha u(x)$

Utility Functions (2)

- Popular examples of homogeneous utility functions:
  - Linear utility function: $u_i(x) = \sum_j a_{ij} x_{ij}$
  - Cobb-Douglas function: $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, $(\sum_j a_{ij} = 1)$
  - Leontief (fixed-proportion) function: $u_i(x) = \min_j a_{ij} x_{ij}$

- General, constant elasticity of substitution form (CES)

$$u_i(x) = \left(\sum_j (a_{ij} x_{ij})^{\rho} \right)^{\frac{1}{\rho}}$$

where $-\infty < \rho < 1, \rho \neq 0$

10

The Fisher Model

The Fisher Model

- Market consisting of
  - n goods sold by <u>one</u> seller
    - total of $q_j > 0$ of good j available
  - m utility maximizing buyers
    - concave utility function $u_i: \mathbb{R}_+^n \to \mathbb{R}_+$
    - Initial endowment $e_i > 0$ of *money*
  - Budget constrain: $\pi \cdot x \leq e_i$

The Fisher Model (2)

- Special case of the exchange economy model
  - initial endowments are proportional

$$w_i = \delta_i w, \qquad \delta_i > 0$$

- If traders have *homogeneous* utility functions
  - => equilibrium can be obtained by solving a convex program (Eisenberg‘s program)

Convex Programs

- Description of a convex optimization problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- Elements:
  - A **convex function** $f(x)$: $\mathbb{R}^n \rightarrow \mathbb{R}$ (to be minimized)
  - **Inequality constraints** $g_i(x) \leq 0$
    - functions g_i are <u>convex</u>
  - **Equality constraints** $h_i(x) = 0$
    - functions h_i are <u>affine</u>

Convex Programs (2)

- Allow to …
  - Minimizing convex functions
  - Maximizing concave functions
  - test feasibility of convex/concave constraints

- There are polynomial algorithms for solving convex programs
  - Ellipsoid method (polynomial, but slow)
  - Interior-point method (polynomial, 1994)

Eisenberg‘s program

- Any optimal solution of the convex program

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \sum_i e_i \log u_i(x_i) \\ \text{subject to} \quad & \sum_i x_{ij} \le q_j \quad \text{for each } j \end{aligned}$$

  on nonnegative variables x_{ij} yields an allocations of goods constituting a market equilibrium.

- Note: no prices involved!

Solving Exchange Economies

Linear Utility Functions

- Suppose that $u_i(x) = \sum_j a_{ij} x_{ij}$ and $w_{ij} > 0$ for each i,j
  - The problem of finding an equilibrium can be written as a finite convex *feasibility problem:*

    Find ψ_j and nonnegative x_{ij} such that

    $$\sum_k a_{ik} x_{ik} \geq a_{ij} \sum_k w_{ik} e^{\psi_k - \psi_j} \quad for\ each\ i,j$$

    $$\sum_i x_i = \sum_i w_i$$

    Any solution to this problem corresponds to an equilibrium obtained by setting $\pi_j = e^{\psi_j}$

CES Utility Functions

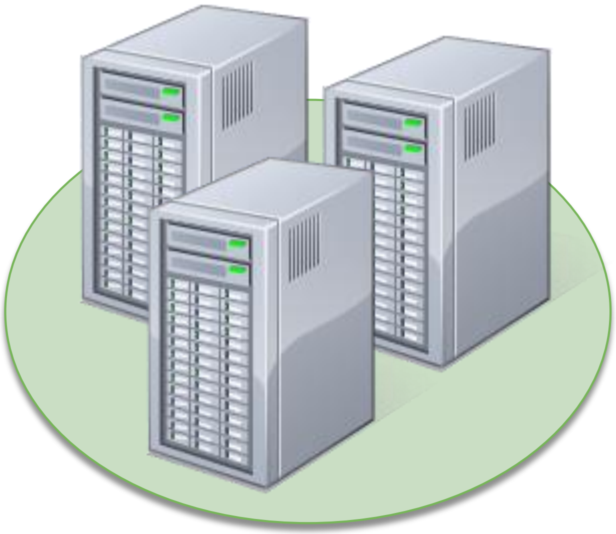
- If $\rho_i > 0$: The Discrete Tâtonnement Process
  - Iterative trial-and-error process
  - delivers approximated solution in polynomial time
- If $-1 \leq \rho_i < 0$: convex *feasibility* problem
  - n inequality constrains of length $O(m)$
- If $\rho_i < -1$ for some trader i, the set of equilibria is no longer convex
  - can be reduced to computing a Nash equilibrium for two-player nonzero sum games
  - => solving Leontief exchange economies is PPAD hard

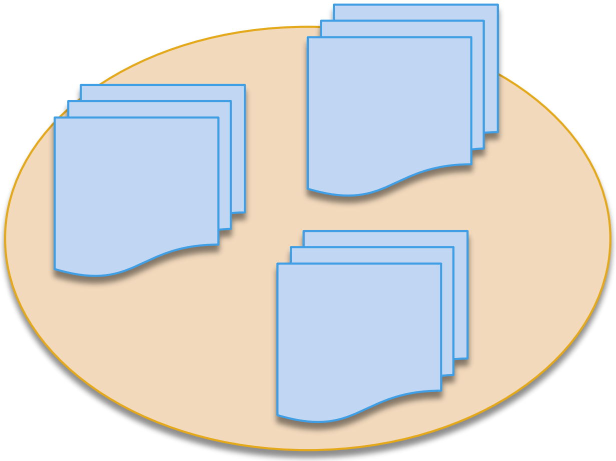
19

Practical Application

Resource Allocation

- The Insieme Runtime:





Resources
HW Threads, Memory, GPU Time

Control Flows
independent, parallel regions

Example Economic Model

- Based on the Fisher Model:
  - goods:
    - Each CPU is a kind of good (hardware threads)
    - The shared memory of each computer
    - Each GPU is a good (time shared)
  - buyers: the parallel regions + one idle process
    - Each region defines its own utility function
    - Each region has an initial endowment (money)
      - Can be used to prioritize the execution of some regions

Problem: Utility Functions

- Linear utility functions
  - would make memory and CPUs exchangeable (!)
- Cobb-Douglas functions
  - same amount of money for each good?
- Leontief (fix-proportion) function
  - Good for product mix
  - Problem of choosing between options …

Solution

- Sum of convex functions is convex!
  - + preserves all other utility function constraints

- Utility function:
  - For each node …
    - => define one fixed-proportion utility function
  - Linear combination of those functions constitutes an adequate utility function!

Conclusion

- Covered Models for Economies
  - Fisher Model
  - Exchange economies
- Definition of Equilibria
  - + Overview on means for computing those
  - + Complexity and Limitations
- A potential application within the Insieme project

25

Thank you for your attention