Selfish Load Balancing

(Seminar of Game Theory)

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1. ***Game Theoretic Job Allocation***
2. Selfish Load Balancing
3. Study of Efficiency of Nash Equilibrium

Task Allocation in Distributed Systems

- Set of Tasks
  - **Independent**
  - Dependent
- Set of Machines
  - **Homogenous (Identical)**
  - **Heterogeneous (Uniformly Related)**
- Task Allocation (Scheduling) Policiy
  - **Static**
  - Dynamic
- Objective Function
  - **Makespan (maximum completion time)**
  - Sum of Execution Time
  - Throughput
  - Cost and etc.

Motivation for Using Game Theory

- Existence of Different Agents
- Different Requirements for Agents
- Selfish Manner of Agents (in many cases)
- Decentralized Scheduling

Study of Job Allocation Game

- Existence of Nash Equilibria
- Complexity of Computing Nash Equilibria
- Efficiency of the Equilibria
  - Price of anarchy (coordination ratio)
  - Price of stability

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Selfish Load Balancing

- **Game**
  - Scheduling in Static Mode
- **Players**
  - Independent Tasks
- **Selfish Approach**
  - Tasks selfishly select the machine with the smallest load
- **Objective Function (Social Cost)**
  - Minimizing the Makespan

Selfish Load Balancing Strategies

- Pure Strategies

- Mixed Strategies

Pure Strategy Selfish Load Balancing (1)

- Set of Tasks

  $[n] = \{1, \ldots, n\}$ with weights $w_1, \ldots, w_n$

- Set of Machines

  $[m] = \{1, \ldots, m\}$ with speeds $s_1, \ldots, s_m$

  on identical machines $s_1 = s_2 = \cdots = s_m = 1$

- Assignment

  $A : [n] \to [m]$

Pure Strategy Selfish Load Balancing (2)

- Load of Machine *j* in Assignment *A*

$$\ell_j = \sum_{\substack{i \in [n] \\ j = A(i)}} \frac{w_i}{s_j}$$

- Cost of Agent *i* Under Assignment *A*

$$c_i^j = \ell_{A(i)}$$

- Social Cost of Assignment *A*

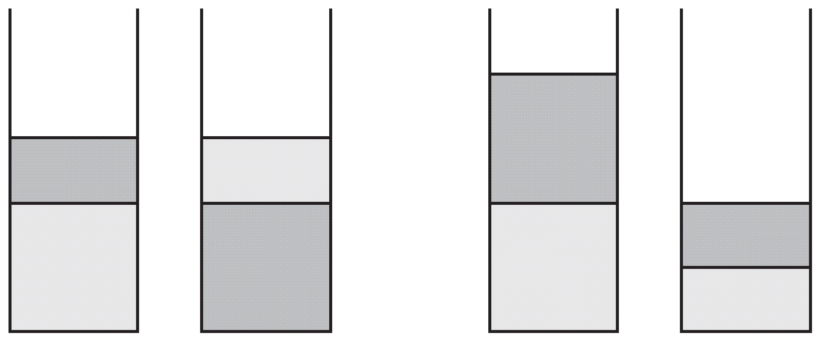
$$\mathrm{cost}(A) = \max_{j \in [m]} (\ell_j)$$

Pure Nash Equilibrium

An assignment A is a pure Nash equilibrium if and only if
$\forall i \in [n] : \forall k \in [m] : c_i^{A(i)} \leq c_i^k.$

An Example of Pure Strategy

$$m = 2, n = 4$$
$$s_1 = s_2 = 1$$
$$w_1 = w_2 = 2, w_3 = w_4 = 1$$



Optimal assignment

$$\ell_1 = \ell_2 = 3$$
$$Cost(A) = 3$$

The worst pure Nash equilibria

$$\ell_1 = 4, \ell_2 = 2$$
$$Cost(A) = 4$$

$$PoA(A) = \frac{4}{3}$$

A Proposition in Selfish Load Balancing

Proposition *Every instance of the load balancing game admits at least one pure Nash equilibrium.*

PROOF:
Each assignment creates a sorted load vector $(\lambda_1, \dots, \lambda_m)$ while λ_j shows load of the machine that has j-th highest load.

If an assignment is not a Nash equilibria, then there is a task i that can perform an improvement step.

We prove the new sorted load vector after each improvement step is lexicographically less that the old sorted load vector.

Before improvement step: $(\lambda_1, \dots, \lambda_{j-1}, \lambda_j, \dots, \lambda_k, \dots, \lambda_m)$, $k > j$

After improvement step: $(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, \dots, \lambda'_m)$

$(\lambda_1, \dots, \lambda_{j-1}, \lambda_j, \dots, \lambda_k, \dots, \lambda_m) > (\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, \dots, \lambda'_m)$

Mixed Strategy Selfish Load Balancing (1)

- Probability of Assigning Task *i* to Machine *j*

$$p_i^j = \mathbb{P}[A(i) = j]$$

- Strategy Profile

$$P = (p_i^j)_{i \in [n], j \in [m]}$$

- An Auxiliary Variable

$$x_i^j = \begin{cases} 1 & if\ A(i) = j \\ 0 & if\ A(i) \neq j \end{cases}$$

Mixed Strategy Selfish Load Balancing (2)

- Expected Load of Machine *j*

$$\mathbb{E}[\ell_j] = \mathbb{E}\left[\sum_{i\in[n]} \frac{w_i\, x_i^j}{s_j}\right] = \sum_{i\in[n]} \frac{w_i\, \mathbb{E}[x_i^j]}{s_j} = \sum_{i\in[n]} \frac{w_i\, p_i^j}{s_j}$$

- Cost of Machine *j* from Point of View of Task *i*

$$c_i^j = \mathbb{E}[\ell_j \mid A(i) = j] = \frac{w_i + \sum_{k\neq i} w_k\, p_k^j}{s_j} = \mathbb{E}[\ell_j] + (1 - p_i^j)\cdot \frac{w_i}{s_j}$$

- Social Cost of Strategy Profile *P*

$$\mathrm{cost}(P) = \mathbb{E}[\mathrm{cost}(A)] = \mathbb{E}\Big[\max_{j\in[m]} (\ell_j)\Big]$$

Mixed Nash Equilibrium

A strategy profile P is a Nash equilibrium if and only if

$$\forall i \in [n] : \forall j \in [m] : p_i^j > 0 \Rightarrow \forall k \in [m] : c_i^j \leq c_i^k .$$

An Example of Mixed Strategy

- Strategy Profile

  $p_i^j = \frac{1}{2}$ for $1 \leq i \leq 4, 1 \leq j \leq 2$

- Expected Load of Machines

$$\mathbb{E}[\ell_j] = \sum_{1 \leq i \leq 4} w_i \, p_i^j = 2 \cdot 2 \cdot \frac{1}{2} + 2 \cdot 1 \cdot \frac{1}{2} = 3$$

- Cost of Machines from Point of View of Tasks

$$c_1^1 = \mathbb{E}[\ell_1] + (1 - p_1^1)\, w_1 = 3 + \frac{1}{2} \cdot 2 = 4,$$

$$c_3^1 = \mathbb{E}[\ell_1] + (1 - p_3^1)\, w_3 = 3 + \frac{1}{2} \cdot 1 = 3.5.$$

$$\left.\begin{matrix} c_1^1 = c_2^1 = c_1^2 = c_2^2 \\ c_3^1 = c_4^1 = c_3^2 = c_4^2 \end{matrix}\right\} \Longrightarrow P \text{ is Nash Equilibria}$$

- Social Cost of Strategy Profile *P*

  There are $2^4 = 16$ different assignments of four tasks to two machines.

  The number of assignments that yield a makespan of 3 is 4, 4 is 6, 5 is 4, and 6 is 2.

$$\text{cost}(P) = \mathbb{E}[\text{cost}(A)] = \frac{1}{16}\,(3 \cdot 4 + 4 \cdot 6 + 5 \cdot 4 + 6 \cdot 2) = 4.25$$

The Main Results

- Mixed equilibria can be worse than the worst pure equilibria.

- Uncoordinated, selfish behavior can lead to suboptimal assignments.

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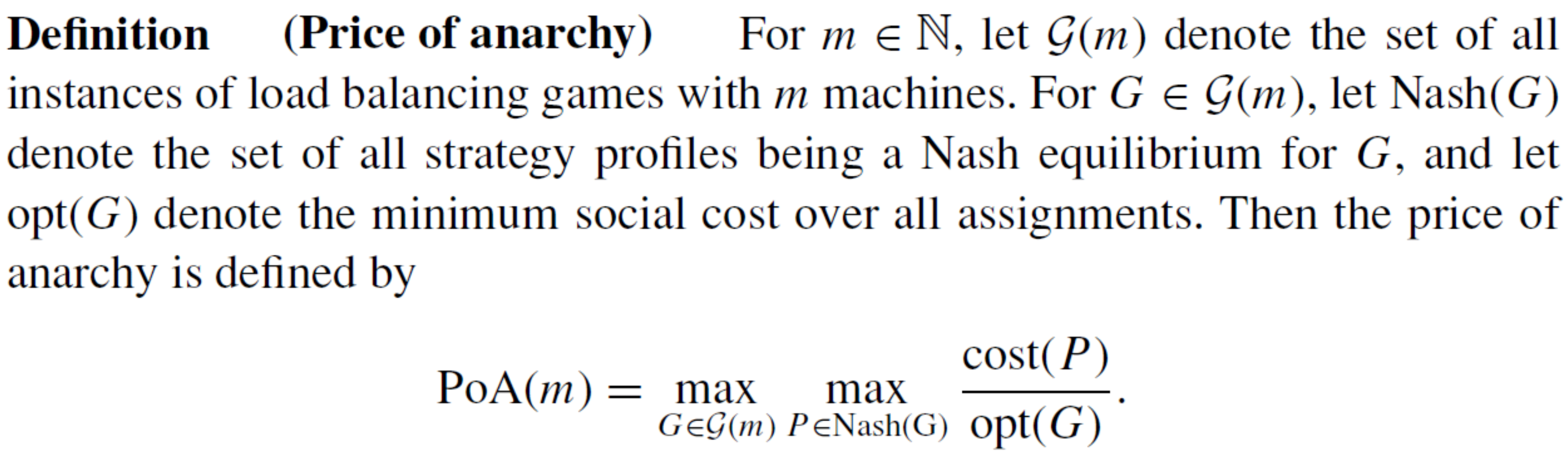
Price of Anarchy in Selfish Load Balancing



Definition **(Price of anarchy)** For $m \in \mathbb{N}$, let $\mathcal{G}(m)$ denote the set of all instances of load balancing games with m machines. For $G \in \mathcal{G}(m)$, let $\mathrm{Nash}(G)$ denote the set of all strategy profiles being a Nash equilibrium for G, and let $\mathrm{opt}(G)$ denote the minimum social cost over all assignments. Then the price of anarchy is defined by

$$\mathrm{PoA}(m) = \max_{G \in \mathcal{G}(m)} \max_{P \in \mathrm{Nash(G)}} \frac{\mathrm{cost}(P)}{\mathrm{opt}(G)}.$$

Importance of Studying PoA

- To quantify the increase of the social cost due to selfish behavior.
- Pure or Mixed Equilibria?
  - Pure Equilibria: In repeatedly improvement steps by agents
  - Mixed Equilibria: In one shot load balancing game

Investigation of Nash Equilibrium in Variant Scenarios

1. Pure Equilibria for Identical Machines
2. Pure Equilibria for Uniformly Related Machines
3. Mixed Equilibria on Identical Machines
4. Mixed Equilibria on Uniformly Related Machines

Investigation of Nash Equilibrium in Variant Scenarios

1. ***Pure Equilibria for Identical Machines***

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3. Mixed Equilibria on Identical Machines

4. Mixed Equilibria on Uniformly Related Machines

Range of PoA

Theorem *Consider an instance G of the load balancing game with n tasks of weight $w_1, \ldots, w_n$ and m identical machines. Let $A : [n] \to [m]$ denote any Nash equilibrium assignment. Then, it holds that*

$$cost(A) \le \left(2 - \frac{2}{m+1}\right) \cdot opt(G).$$

Proof Let j^* be a machine with the highest load under assignment A

let i^* be a task of smallest weight assigned to this machine

there are at least two tasks assigned to machine j^* as, otherwise, $\text{cost}(A) = \text{opt}(G)$

Thus $w_{i^*} \le \frac{1}{2}\text{cost}(A)$

Suppose there is a machine $j \in [n] \setminus \{j^*\}$ with load less than $\ell_{j^*} - w_{i^*}$

$$\ell_j \ge \ell_{j^*} - w_{i^*} \ge \text{cost}(A) - \frac{1}{2}\text{cost}(A) = \frac{1}{2}\text{cost}(A)$$

$$\text{opt}(G) \ge \frac{\sum_{i\in[n]} w_i}{m} = \frac{\sum_{j\in[m]} \ell_j}{m} \ge \frac{\text{cost}(A) + \frac{1}{2}\text{cost}(A)(m-1)}{m} = \frac{(m+1)\text{cost}(A)}{2m}$$

$$\text{cost}(A) \le \frac{2m}{m+1} \cdot \text{opt}(G) = \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

Generalizing the Numerical Example

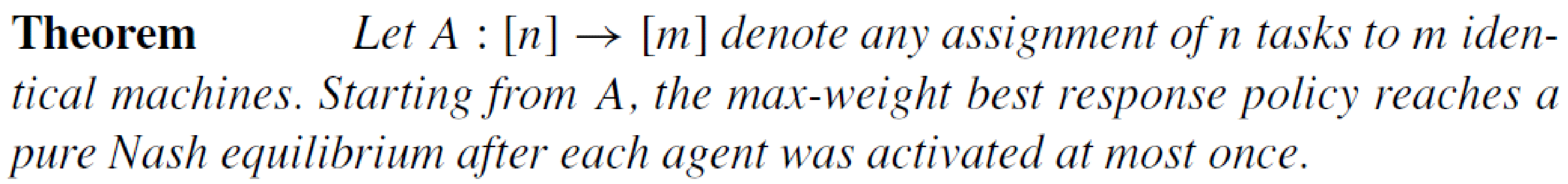
a price of anarchy of $\frac{4}{3} = 2 - \frac{2}{m+1}$, for $m = 2$

the load balancing game with m identical machines and $2m$ tasks that has a Nash equilibrium assignment $A : [n] \to [m]$ with

$$\text{cost}(A) = \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

Thus the upper bound on the price of anarchy is tight.

Convergence Time of Best Responses



Theorem *Let $A : [n] \to [m]$ denote any assignment of n tasks to m identical machines. Starting from A, the max-weight best response policy reaches a pure Nash equilibrium after each agent was activated at most once.*

Satisfied Agent

- If agent cannot reduce its cost by unilaterally moving its task to another machine.

Max- weight best response policy

- Activates the agents one after the other
- Always activates an agent with maximum weight among the unsatisfied agents
- An activated agent moves its task to the machine with minimum load

Investigation of Nash Equilibrium in Variant Scenarios

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2. ***Pure Equilibria for Uniformly Related Machines***
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Range of PoA

Theorem *Consider an instance G of the load balancing game with n tasks of weight* $w_1, \ldots, w_n$ *and m machines of speed* $s_1, \ldots, s_m$. *Let* $A : [n] \to [m]$ *denote any Nash equilibrium assignment. Then, it holds that*

$$cost(A) = \mathcal{O}\left(\frac{\log m}{\log \log m}\right) \cdot opt(G).$$

An Algorithm for Computing Pure Equilibria

Theorem *The LPT algorithm computes a pure Nash equilibrium for load balancing games on uniformly related machines.*

LPT (Largest Processing Time)

Inserting the tasks in a non-increasing order of weights

Assigning each task to a machine that minimizes the cost of the task at its insertion time.

Investigation of Nash Equilibrium in Variant Scenarios

1. Pure Equilibria for Identical Machines

2. Pure Equilibria for Uniformly Related Machines

3. ***Mixed Equilibria on Identical Machines***

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Range of PoA

Theorem *Consider an instance G of the load balancing game with n tasks of weight $w_1, \ldots, w_n$ and m identical machines. Let $P = (p_i^j)_{i \in [n], j \in [m]}$ denote any Nash equilibrium strategy profile. Then, it holds that*

$$cost(P) = \mathcal{O}\left(\frac{\log m}{\log \log m}\right) \cdot opt(G).$$

Investigation of Nash Equilibrium in Variant Scenarios

1. Pure Equilibria for Identical Machines

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4. ***Mixed Equilibria on Uniformly Related Machines***

Range of PoA

Theorem *Consider an instance G of the load balancing game with n tasks of weight $w_1, \ldots, w_n$ and m machines of speed $s_1, \ldots, s_m$. Let P be any Nash equilibrium strategy profile. Then, it holds that*

$$cost(P) = \mathcal{O}\left(\frac{\log m}{\log \log \log m}\right) \cdot opt(G).$$

Summary

- Price of Anarchy for pure strategy on identical machine is maximum 2.
- In the other case it is a slightly growing sub logarithmic function in the number of machines.
- Computing the equilibrium in identical machine from any initial sequence convergences fast.
- Complexity of computing the equilibrium in the case of uniformly related machine it is not known.
- LPT algorithm in uniformly related machine can compute a Nash equilibria efficiently.

Some New Roadmaps

- Making the more realistic models, more realistic cost functions or other ways to define the social cost.
- Building a distributed system that does not suffer from selfish behavior but might even exploit the selfishness of the agents.
- Finding other way but improvement steps that quickly converge to a Nash equilibrium or approximate Nash equilibrium.

Thank you for your attention.