1. Consider the following propositional formulas

$$A := (p \to q) \land (q \to \neg p) \to p$$
$$B := (p \land q \to r) \to (p \to (q \to r))$$

   a) Show that formula A is satisfiable by giving a satisfying assignment. (4 pts)
   b) Show that formula B is valid, by giving a proof in propositional resolution. (6 pts)

2. Consider the following sentences:

   ① A dragon is happy if its mother or its father is happy.
   ② A dragon can spit fire, if one of its ancestors can spit fire.
   ③ The mother or the father of a dragon are ancestors.
   ④ An ancestor of an ancestor is an ancestor.
   ⑤ There exists a blue dragon that cannot spit fire.

   a) For each of the sentences above, give a first-order formula that formalises the sentence. Use therefore *only* the following constants, functions and predicates:
      – constants: blue, red
      – functions: $\mathsf{colour}(x)$
      – predicates: $\mathsf{Dragon}(x)$, $\mathsf{Happy}(x)$, $\mathsf{Spitfire}(x)$, $\mathsf{Father}(x, y)$, $\mathsf{Mother}(x, y)$, $\mathsf{Ancestor}(x, y)$, $=$

      Note that the predicates $\mathsf{Father}(x, y)$, $\mathsf{Mother}(x, y)$, $\mathsf{Ancestor}(x, y)$ are to be interpreted as y is father of x, y is mother of x, y is ancestor of x, respectively. (5 pts)
   b) Show that your formalisation is satisfiable. (3 pts)

3. Consider first-order logic *without* equality. Let $\mathcal{I} = (\mathcal{A}, \ell)$ be an interpretation and F a formula. Suppose we restrict the logical symbols in any first-order language to $\neg$, $\land$, and $\exists$. Moreover, suppose that we do not distinguish (in the notation) between free and bound variables. Then we attempt to define the satisfaction relation $\mathcal{I} \models F$ as follows:
   – $\mathcal{I} \models P(t_1, \dots, t_n)$ iff $(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) \in P^{\mathcal{I}}$
   – $\mathcal{I} \models \neg F$ iff $\mathcal{I} \not\models F$.
   – $\mathcal{I} \models F \land G$ iff $\mathcal{I} \models F$ and $\mathcal{I} \models G$.
   – $\mathcal{I} \models \exists x F(x)$ iff there exists $a \in A$ (A the domain of $\mathcal{A}$) and $\mathcal{I}\{x \mapsto a\} \models F(x)$ holds.

   a) Is this definition correct? Explain your answer. (4 pts)
   b) Regardless of the correctness of the definition, the definition is incomplete for the language of first-order logic without equality as defined in the lecture. Explain why. (3 pts)

c) Explain how this definition can be made complete (not necessarily correct). (3 pts)

4. Consider the following first-order formulas (with =) with individual constants a, b, and predicate constants P, Q, and R:

$$C := \exists x \forall y \exists z \forall u \exists v (\mathsf{P}(x) \lor \neg \mathsf{Q}(y,z) \lor \mathsf{R}(u,v))$$
$$D := \forall x\ (\mathsf{P}(x) \to \exists y \forall z (\mathsf{P}(z) \lor \mathsf{Q}(x,y) \to \forall w \mathsf{R}(x,w)))$$
$$E := \mathsf{a} = \mathsf{b} \land \exists x \mathsf{Q}(\mathsf{a},x) \to \exists x \mathsf{Q}(\mathsf{b},x)$$

a) Give the SNF of formula C. (3 pts)

b) Give the SNF of formula D. (3 pts)

a) Use paramodulation for first-order to show that formula E is valid. (6 pts)

5. Determine whether the statements on the answer sheet are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

- Consider propositional logic. Then $A_1, \ldots, A_n \models B$, asserts that $\mathsf{v}(B) = \mathsf{T}$, whenever there exists $i \in \{1, \ldots, n\}$ such that $\mathsf{v}(A_i) = \mathsf{T}$, for any assignment v.
- Natural deduction for propositional logic is sound and complete.
- Let $\mathcal{A}$, $\mathcal{B}$ be first-order structures such that $\mathcal{A} \cong \mathcal{B}$ and let ℓ be an environment. Then for every formula F we have $(\mathcal{A}, \ell) \models F$ iff $(\mathcal{B}, \ell) \models F$.
- If formula G is obtained from formula F on replacing a subformulas A by an equivalent formula B then F and G are equivalent.
- For any formula F there exists a formula G such that G does neither contain individual or function constants nor equality and $F \approx G$.
- The set S of all consistent set of formulas has the satisfaction properties.
- Let $\mathcal{G}$ be a countable set of formulas, if $\mathcal{G}$ is consistent, then $\mathcal{G}$ has only uncountable models.
- For directed graphs, reachability is expressible as existential, first-order formula.
- There exists no satisfiable set of first-order sentences $\mathcal{G}$, such that there exists no Herbrand model of $\mathcal{G}$.
- For any first-order sentence F there exists a set of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $F \approx \forall x_1 \ldots \forall x_n (C_1 \land \cdots \land C_m)$.