

Functional Programming

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λ -Calculus

λ -Terms

$$t ::= \overbrace{x}^{\text{Variable}} \mid \underbrace{(\lambda x.t)}_{\text{Abstraction}} \mid \overbrace{(t t)}^{\text{Application}}$$

Example

| | | |
|-------------------|-----------------------------|---------------------------------------|
| $x y$ | $(x y)$ | <i>"x applied to y"</i> |
| $\lambda x.x$ | $(\lambda x.x)$ | <i>"lambda x to x"</i> |
| $\lambda xy.x$ | $(\lambda x.(\lambda y.x))$ | <i>"lambda x y to x"</i> |
| $\lambda x.x x$ | $(\lambda x.(x x))$ | <i>"lambda x to (x applied to x)"</i> |
| $(\lambda x.x) x$ | $((\lambda x.x) x)$ | <i>"(lambda x to x) applied to x"</i> |

λ -Calculus (cont'd)

β -Reduction

the term s (β -)reduces to the term t in one step, i.e.,

$$\underbrace{s \rightarrow_{\beta} t}_{(\beta\text{-})\text{step}}$$

iff there exist context C and terms u, v s.t.

$$s = C[(\lambda x.u) v] \quad \text{and} \quad t = C[u\{x/v\}]$$

Example

$$K \stackrel{\text{def}}{=} \lambda xy.x$$

$$I \stackrel{\text{def}}{=} \lambda x.x$$

$$\Omega \stackrel{\text{def}}{=} (\lambda x.x x) (\lambda x.x x)$$

This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, monads, . . .

Booleans and Conditionals

OCaml

- ▶ `true`
- ▶ `false`
- ▶ `if b then t else e`

λ -Calculus

- ▶ `true` $\stackrel{\text{def}}{=} \lambda xy.x$
- ▶ `false` $\stackrel{\text{def}}{=} \lambda xy.y$
- ▶ `if` $\stackrel{\text{def}}{=} \lambda xyz.x y z$

Example

$$\text{if true } t e \rightarrow_{\beta}^+ \text{true } t e \rightarrow_{\beta}^+ t$$

$$\text{if false } t e \rightarrow_{\beta}^+ \text{false } t e \rightarrow_{\beta}^+ e$$

Natural Numbers

Definition

$$s^0 t \stackrel{\text{def}}{=} t$$

$$s^{n+1} t \stackrel{\text{def}}{=} s (s^n t)$$

OCaml vs. λ -Calculus

$$0 \quad \bar{0} \stackrel{\text{def}}{=} \lambda f x. x$$

$$1 \quad \bar{1} \stackrel{\text{def}}{=} \lambda f x. f x$$

$$n \quad \bar{n} \stackrel{\text{def}}{=} \lambda f x. f^n x$$

$$(+) \quad \text{add} \stackrel{\text{def}}{=} \lambda m n f x. m f (n f x)$$

$$(*) \quad \text{mul} \stackrel{\text{def}}{=} \lambda m n f. m (n f)$$

$$(**) \quad \text{exp} \stackrel{\text{def}}{=} \lambda m n. n m$$

Example

$$\text{add } \bar{1} \bar{1} \rightarrow_{\beta}^* \bar{2}$$

Pairs

OCaml vs. λ -Calculus

| | |
|----------------------------------|---|
| <code>fun x y -> (x,y)</code> | <code>pair</code> $\stackrel{\text{def}}{=} \lambda xyf.f x y$ |
| <code>fst</code> | <code>fst</code> $\stackrel{\text{def}}{=} \lambda p.p \text{ true}$ |
| <code>snd</code> | <code>snd</code> $\stackrel{\text{def}}{=} \lambda p.p \text{ false}$ |

Example

$$\text{fst (pair } \bar{m} \bar{n}) \rightarrow_{\beta}^* \bar{m}$$

Lists

OCaml vs. λ -Calculus

| | |
|---------------------------------|---|
| <code>::</code> | $\text{cons} \stackrel{\text{def}}{=} \lambda xy. \text{pair } \text{false} (\text{pair } x \ y)$ |
| <code>hd</code> | $\text{hd} \stackrel{\text{def}}{=} \lambda z. \text{fst } (\text{snd } z)$ |
| <code>tl</code> | $\text{tl} \stackrel{\text{def}}{=} \lambda z. \text{snd } (\text{snd } z)$ |
| <code>[]</code> | $\text{nil} \stackrel{\text{def}}{=} \lambda x. x$ |
| <code>fun x -> x = []</code> | $\text{null} \stackrel{\text{def}}{=} \text{fst}$ |

Example

$$\text{null nil} \rightarrow_{\beta}^* \text{true}$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                  else 1 + length(tl x)
```

λ -Calculus

$$\text{length} \stackrel{\text{def}}{=} Y (\lambda f x. \text{if } (\text{null } x) \bar{0} (\text{add } \bar{1} (f (\text{tl } x))))$$

Definition (Y-combinator)

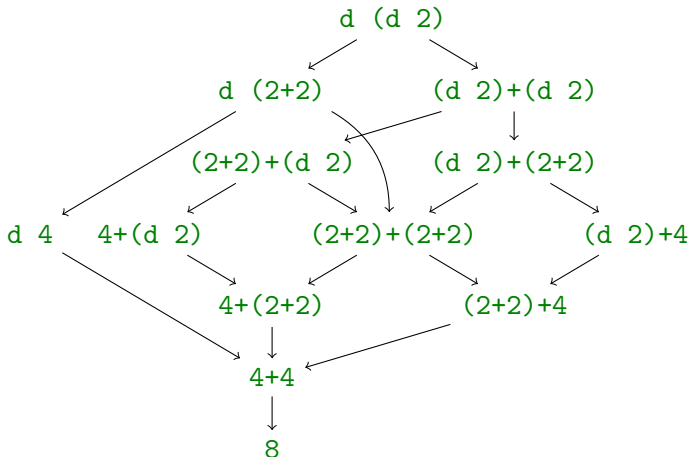
$$Y \stackrel{\text{def}}{=} \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

Y has **fixed point property**, i.e., for all $t \in \mathcal{T}(\mathcal{V})$

$$Y t \leftrightarrow^* t (Y t)$$

Example

- ▶ consider `let d x = x + x`
- ▶ the term `d (d 2)` can be evaluated as follows (9 possibilities)



Strategies

Strategy

- ▶ fixes evaluation order
- ▶ examples: **call-by-value** and **call-by-name**

Example

let $d\ x = x + x$

- ▶ call-by-value:

$$\begin{aligned} d\ (d\ 2) &\rightarrow d\ (2+2) \\ &\rightarrow d\ 4 \\ &\rightarrow 4 + 4 \\ &\rightarrow 8 \end{aligned}$$

- ▶ call-by-name:

$$\begin{aligned} d\ (d\ 2) &\rightarrow (d\ 2) + (d\ 2) \\ &\rightarrow (2+2) + (d\ 2) \\ &\rightarrow 4 + (d\ 2) \\ &\rightarrow 4 + (2+2) \\ &\rightarrow 4 + 4 \\ &\rightarrow 8 \end{aligned}$$

(Leftmost) Innermost Reduction

- ▶ always reduce leftmost innermost redex

Definition

redex t of term u is **innermost** if it does not contain a redex as **proper** subterm, i.e.,

$$\nexists s \in \text{Sub}(t) \text{ s.t. } s \neq t \text{ and } s \text{ is a redex}$$

Example

Consider $t = (\lambda x. (\lambda y. y) x) z$

- ▶ $(\lambda y. y) x$ is innermost redex
- ▶ $(\lambda x. (\lambda y. y) x) z$ is redex, but not innermost

(Leftmost) Outermost Reduction

- ▶ always reduce leftmost outermost redex

Definition

redex t of term u is **outermost** if it is not a **proper** subterm of some other redex in u , i.e.,

$$\nexists s \in \text{Sub}(u) \text{ s.t. } s \text{ is a redex and } t \in \text{Sub}(s) \text{ and } s \neq t$$

Example

Consider $t = (\lambda x. (\lambda y. y) x) z$

- ▶ $(\lambda x. (\lambda y. y) x) z$ is outermost redex
- ▶ $(\lambda y. y) x$ is redex, but not outermost

Call-by-Value

- ▶ use innermost reduction
- ▶ corresponds to strict (or eager) evaluation, e.g., OCaml
- ▶ slight modification: only reduce terms that are not in WHNF

Definition (Weak head normal form)

term t is in **weak head normal form** (*WHNF*) iff

$$t \neq u \ v$$

Example (WHNF)

$\lambda x.x$ ✓ $(\lambda x.x) \ y$ ✗ $(\lambda x.x) \ y \ z$ ✗ $\lambda x.(\lambda y.y) \ x$ ✓ $x \ x$ ✗

Call-by-Name

- ▶ use outermost reduction
- ▶ corresponds to lazy evaluation (without memoization), e.g., Haskell
- ▶ slight modification: only reduce terms that are not in WHNF

λ Tree Tool

developed by Stefan Widerin in [bachelor project](#)

λ -Terms

$$t ::= x \mid (\lambda x. t) \mid (t t)$$

Conventions

- ▶ nested abstractions use spaces to separate variable names, e.g.,

$$\begin{aligned} \lambda xy. x & \quad \lambda x y. x \\ \lambda x_1. y & \quad \lambda x_1. y \end{aligned}$$

Result

Animator

- ▶ animation of β -steps
- ▶ innermost/outermost
- ▶ leftmost/rightmost

Output

- ▶ reduction sequence to NF
- ▶ innermost/outermost
- ▶ leftmost/rightmost