First Exam
Gödel's Incompleteness Theorem, LVA 703919

February 8, 2012

Name: **Number of Studies**: **Studentnumber**:

The exam consists of 5 exercises with a total of 50 points. Please fill out your name and credentials *before* you start the exam.

1	2	3	4	5	Sum

1. Suppose $\mathcal{L}$ is a correct system such that the following two conditions hold, where P denotes the set of Gödel numbers of the provable sentences of $\mathcal{L}$.
   - The set P^* is *expressible* in $\mathcal{L}$.
   - For any predicate H, there is a predicate H' such that for every n, the sentence $H'(n)$ is provable in $\mathcal{L}$ iff $H(n)$ is refutable in $\mathcal{L}$.

   Show that $\mathcal{L}$ is incomplete. (10 pts)

2. Consider a set $A \supseteq R^*$ such that $A \cap P^* = \varnothing$ and A is *representable* in $\mathcal{L}$. Here P is defined as above, and R denotes the set of Gödel numbers of refutable sentences of $\mathcal{L}$.
   a) Show that $\mathcal{L}$ is consistent. (2 pts)
   b) Show that $\mathcal{L}$ is incomplete (10 pts)

3. Consider the language $\mathcal{L}$ of PA. Suppose the expression $\mathsf{formula}(x)$ shall denote that E_x is a formula. Explain informally (but in precise terms) what is necessary to give an arithmetisation of $\mathsf{formula}(x)$.

   You may use any part of the arithmetisation you remember, in particular you can assume that $\ulcorner E \urcorner$ denotes the *Gödel number* of E. (8 pts)

4. Recall the following:
   - A relation $R(x,y)$ is recursive if $R(x,y)$ and $\sim R(x,y)$ is Σ_1.
   - Similarly, a function f is recursive if its graph is recursive.

   Show that for any recursive relation $R(x,y)$ and any recursive function $f(x)$, the following relation is recursive:

   $$Q(x) := (\exists y \leqslant f(x))R(x,y) \quad .$$

   (10 pts)

5. Determine whether the following statements are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

   (See next page.)

statement	yes	no
Let $\mathcal{S}$ be any extension of PA. If $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete.	☐	☐
if $\mathcal{L}$ is correct and if $(\sim P)^*$ is expressible in $\mathcal{L}$, then $\mathcal{L}$ is complete.	☐	☐
The set T of Gödel numbers of the true arithmetic sentences is arithmetic.	☐	☐
The relation $x^y = z$ is Σ_0.	☐	☐
All true Σ_0-sentences of Robinson's Q are provable in PA.	☐	☐
Suppose the system $\mathcal{S}$ is consistent and all true Σ_0-sentences are provable. This does not imply that all provable Σ_0-sentences are true.	☐	☐
Suppose $F(v_1)$ separates A from B in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $F(v_1)$ represents some superset of A disjoint from B.	☐	☐
Every extension $\mathcal{S}$ of Ω_4, Ω_5 in which all Σ_1-relations are enumerable is a Rosser system.	☐	☐
Rosser's theorem states that every ω-consistent extension of Ω_4, Ω_5 in which all Σ_1-sets are enumerable must be incomplete.	☐	☐
Suppose PA is consistent. Then consistency of PA is not provable in any extension of PA.	☐	☐