1. *Solution.* Let $H(v_1)$ express the set P^* and let $H'(v_1)$ be the predicate according to the assumption. Furthermore we write $\ulcorner E \urcorner$ for the Gödel number of E. We observe the following:

   - $H(\overline{n})$ holds iff $n \in P^*$.
   - $H'(\overline{n})$ is provable iff $H(\overline{n})$ is refutable.
   - If $h' := \ulcorner H'(v_1) \urcorner$, then $H(h')$ holds iff $H(h')$ is refutable.

   Now, suppose $H(h')$ is provable, thus $H(h')$ holds by correctness and then $H(h')$ is refutable, which is a contradiction. Similarly, if $H(h')$ is refutable, then $H(h')$ holds, which contradicts correctness. Hence $\mathcal{L}$ is incomplete. □

2. *Solution.* It is easy to see that the assumptions imply that $\mathcal{L}$ is consistent. Hence, we only give the proof of b). Suppose $F(v_1)$ represents A and $f := \ulcorner F(v_1) \urcorner$. Then we have

   - $F(f)$ is provable iff $f \in A$.
   - $F(f)$ is provable iff $f \in P^*$.

   Hence we obtain $f \in P^*$ iff $f \in A$. Thus $f \notin A$ and $f \notin P^*$. From this observation it follows that $F(f)$ is neither provable, nor refutable. □

3. *Solution.* See §3 in Chapter 2 of Smullyan's book. □

4. *Solution.* By assumption, there exists Σ_0-formulas $S_0(v_1, v_2, v_3)$ and $S_1(v_1, v_2, v_3)$ such that we have the following:

   $$R(m,n) \text{ holds} \leftrightarrow \exists z S_0(\overline{m}, \overline{n}, z)$$
   $$R(m,n) \text{ does not hold} \leftrightarrow \exists z S_1(\overline{m}, \overline{n}, z)$$

   Moreover, there exists a formula $G(v_1, v_2, v_3)$ such that $f(m) = y$ iff $\exists z G(\overline{m}, \overline{n}, z)$. We show that $Q(x)$ is Σ_1 and that $\sim Q(x)$ is Σ_1 as follows:

   $$Q(m) \text{ holds} \leftrightarrow \exists z \exists v \exists w\, (G(\overline{m}, z, v) \land (\exists y \leqslant z) S_0(\overline{m}, y, w))$$
   $$Q(m) \text{ does not holds} \leftrightarrow \exists z \exists v \exists w\, (G(\overline{m}, z, v) \land (\forall y \leqslant z)(\exists u \leqslant w) S_1(\overline{m}, y, u))$$

   Note that for the second formula the existential quantifier bounding w in $S_1(\overline{m}, y, u)$ cannot be directly move in front of the bounded quantifer $(\forall y \leqslant z)$. Hence the need for this seemingly artificial bounded quantifier $(\exists u \leqslant w)$, cf. Chapter 4, Section 2 in Smullyan's book.

   Clearly the formulas to the right are equivalent to Σ_1-sentences. Hence the claim follows. □

5.

Solution.

statement	**yes**	**no**
Let $\mathcal{S}$ be any extension of PA. If $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete.	✓	
if $\mathcal{L}$ is correct and if $(\sim P)^*$ is expressible in $\mathcal{L}$, then $\mathcal{L}$ is complete.		✓
The set T of Gödel numbers of the true arithmetic sentences is arithmetic.		✓
The relation $x^y = z$ is Σ_0.		✓
All true Σ_0-sentences of Robinson's Q are provable in PA.	✓	
Suppose the system $\mathcal{S}$ is consistent and all true Σ_0-sentences are provable. This does not imply that all provable Σ_0-sentences are true.		✓
Suppose $F(v_1)$ separates A from B in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $F(v_1)$ represents some superset of A disjoint from B.	✓	
Every extension $\mathcal{S}$ of Ω_4, Ω_5 in which all Σ_1-relations are enumerable is a Rosser system.	✓	
Rosser's theorem states that every ω-consistent extension of Ω_4, Ω_5 in which all Σ_1-sets are enumerable must be incomplete.		✓
Suppose PA is consistent. Then consistency of PA is not provable in any extension of PA.		✓

□