Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

Winter 2011

Organisation

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Time and Place

- Thursday, 9:15–10:45, SR 12

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Schedule

week 1	October 6	week 8	November 24
week 2	October 13	week 9	December 1
week 3	October 20	week 10	December 15
week 4	October 27	week 11	January 12
week 5	November 3	week 12	January 19
week 6	November 10	week 13	January 26
week 7	November 17	first exam	February 2

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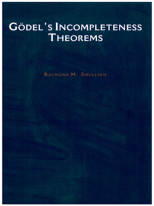
Question

shift lecture, due to overlap with Sequenzanalyse?

Literature

- Raymond M. Smullyan
  Gödel's Incompleteness Theorems
  Oxford Logic Guides, 1992



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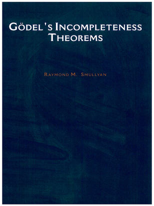

This introduction to Gödel's incompleteness theorems is written for the general mathematician, philosopher, computer scientist and any other curious reader who has at least nodding acquaintance with the symbolism of first-order logic (...) and who can recognize the logical validity of a few elementary formulas.

Homework & Exam

1. officially there are no exercises as this course is labelled VO
2. however, it improves the understanding, if the exercises in the book are studied
3. the homework assignments will be discussed weekly in the lecture
4. participation can only positively influence the final grade

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transparencies and homework will be available from IP starting with 138.232 after the lecture; (almost) every week homework assignments will be provided, to be discussed in the lecture.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarskis's theorems, undecidable sentences of $\mathcal{L}$

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Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

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Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

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Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Introduction

Definition

1. a theory T (or an axiom system) in a first-order language $\mathcal{L}$, is a set of sentences of $\mathcal{L}$ that is closed under the consequence relation
2. $F \in T$ is called a theorem
3. T is complete if for every sentence F of $\mathcal{L}$ either $F \in T$ or $\neg F \in T$
4. a theory T is axiomatisable, if there exists a finite set of schemata such that T consists of all instances of these schemata
5. a theory T' is an extension of a theory T if $T \subseteq T'$

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Remark

for the time being, it is not essential to know the axioms and inferences of PA; we will present many axiomatisation of arithmetic during the course

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Theorem (Gödel's First Incompleteness Theorem)

Let S *be an extension of* PA*. If* S *is consistent, then* S *is incomplete*

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Theorem (Gödel's Second Incompletness Theorem)

If S *is consistent, then the consistency of* S *is not provable in* S

In Gödel's own words

The development of mathematics in the direction of greater precision has led to large areas of it being formalized, so that proofs can be carried out according to a few mechanical rules. The most comprehensive formal systems to date are, on the one hand, the Principia Mathematica by Whitehead and Russel and, on the other hand, the Zermelo-Fraenkel system of axiomatic set theory. [...]

It would seem reasonable [...] to surmise that these axioms and rules of inference are sufficient to decide all mathematical questions which can be formulated in the system concerned.

In what follows it will be shown that this is not the case, but rather that, in both the cited systems, there exist relatively simple problems of the theory of ordinary whole numbers which cannot be decided on the basis of the axioms.

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versus

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Gödel's 2nd incompleteness theorem (almost) buried Hilbert's program

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- Voevodsky's work is in the intersection of algebraic geometry with algebraic topology
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complementing Hilbert's view Voevoedsky seems to have started a foundation program aimed at the "working mathematician"

> *I am working full time on new foundations and the related type theoretic formalization which should finally make it possible for the working mathematicians to use proof development software.*

Discussion on FOM

...

- WV: *Consistency on the other hand is not an interesting problem since it has been shown by Gödel to be impossible to proof.*
- HF: *How would you assess the various proofs of the consistency of PA that we now have, for your purposes?* [...]
- WV: *Well, that would contradict Gödel's result.*
- HF: *Apparently, you think it worthwhile to spend some of your time to prove the inconsistency of PA*. [...] *Actually, working on giving a consistency proof of PA within PA is actually exactly the same problem, by Gödel.*
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- WV: *I do not really work on it. At least not actively.*

the lecture that started the above discussion is available online

`http://video.ias.edu/voevodsky-80th`

Gödelian Puzzles

Puzzle

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5. a sentence is an expression of one the following forms

$$\begin{aligned} \mathrm{P}(X) \\ \mathrm{PN}(X) \\ {\sim}\mathrm{P}(X) \\ {\sim}\mathrm{PN}(X) \end{aligned}$$

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4. the norm of an expression X is $X(X)$
5. a sentence is an expression of one the following forms

$\mathsf{P}(X)$	is true if X is printable
$\mathsf{PN}(X)$	is true if norm of X is printable
$\sim\mathsf{P}(X)$	is true if X is not printable
$\sim\mathsf{PN}(X)$	is true if norm of X is not printable

Puzzle (cont'd)

6. assume M_1 is accurate: all sentences printed are true

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Example

- if M_1 ever prints $\mathsf{P}(X)$, then X is eventually printed
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if X is printable, is then $\mathrm{P}(X)$ printable?

Answer

not necessarily; if X is printable, then $\mathrm{P}(X)$ is true, but M_1 need not print all true sentences

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not necessarily; if X is printable, then $\mathrm{P}(X)$ is true, but M_1 need not print all true sentences

Question ①

is it possible that M_1 can print all true sentences?

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2. the Gödel number $\ulcorner X \urcorner$ of X is the concatenation of the Gödel numbers of the symbols
3. the norm of an expression X is $X\ulcorner X \urcorner$
4. a sentence is an expression of form (X is number in binary)

$\mathsf{P}X$	X is Gödel number of printable expression
$\mathsf{PN}X$	X is Gödel number of norm of printable expression
$\sim\mathsf{P}X$	$\neg\mathsf{P}X$
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Answer

① the answer is no

- consider the sentence $\sim\mathrm{PN}(\sim\mathrm{PN}) =: F$
- F is true iff the norm of $\sim\mathrm{PN}$ is not printable
- the norm of $\sim\mathrm{PN} = F$
- hence F is true iff F is not printable
- assume F is false, thus printable; this contradicts accuracy of M_1
- hence F is true, but not printable

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- consider the sentence $\sim$PN101001000