Gödel's Incompleteness Theorem

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Homework

- Chapter V, Exercise 9
- Chapter V, Exercise 10
- Chapter V, Exercise 11
- Chapter V, Exercise 12
- Chapter VI, Exercise 3 (postpone)
- Chapter VI, Exercise 4 (postpone)

Summary of Last Lecture

Definition (Q)

$$
\begin{aligned}
N_1: &\quad v_1' = v_2' \to v_1 = v_2 \\
N_2: &\quad \overline{0} \neq v_1' \\
N_3: &\quad (v_1 + \overline{0}) = v_1 \\
N_4: &\quad (v_1 + v_2') = (v_1 + v_2)' \\
N_5: &\quad (v_1 \cdot \overline{0}) = \overline{0} \\
N_6: &\quad (v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1) \\
N_7: &\quad (v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0}) \\
N_8: &\quad (v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \vee v_1 = v_2') \\
N_9: &\quad (v_1 \leqslant v_2) \vee (v_2 \leqslant v_1)
\end{aligned}
$$

let Q_0 be Q without the axiom N_9

Definition (R)

$$\Omega_1: \quad \overline{m} + \overline{n} = \overline{k} \quad \text{if } m + n = k$$

$$\Omega_2: \quad \overline{m} \cdot \overline{n} = \overline{k} \quad \text{if } m \cdot n = k$$

$$\Omega_3: \quad \overline{m} \neq \overline{n} \quad \text{if } m \neq n$$

$$\Omega_4: \quad v_1 \leqslant \overline{n} \leftrightarrow (v_1 = \overline{0} \vee \cdots \vee v_1 = \overline{n})$$

$$\Omega_5: \quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1$$

let R_0 be R without the schema Ω_5

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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Lemma

R_0 is Σ_0-complete

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Proof.

it suffices to show the following properties:

- D_1 $\forall$ true atomic Σ_0-sentence A, A is provable
- D_2 $\forall m, n$: $m \neq n$: $\mathcal{S} \vdash \overline{m} \neq \overline{n}$
- D_3 $\forall$ variable w, $\forall n \in \mathbb{N}$:

$$\mathcal{S} \vdash w \leqslant \overline{n} \rightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n})$$

we consider the three properties:

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- D_3 follows from axiom schemata Ω_4

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on the whiteboard ∎

Corollary

R is a subsystem of Q

Σ_0-completeness of PA

Theorem

the systems R_0, R, Q_0, Q, and PA *are Σ_0-complete*

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a system $\mathcal{S}$ is Σ_0-complete if all true Σ_0-sentences are provable in $\mathcal{S}$

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Recall Theorem ②

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Recall Theorem ②

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thus finally, we have proved:

Theorem

if PA is ω-consistent, then it is incomplete

Midterm Quiz

- a sentence S is decidable in $\mathcal{S}$ if it either provable or refutable
- $\mathcal{S}$ is complete if $\forall$ sentences S, S is decidable; otherwise $\mathcal{S}$ is incomplete

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consider the following

Definition

let $\mathcal{S}$ be an arbitrary axiom system (aka theory) and S a sentence

- we write $\mathcal{S} \models S$ if for all interpretation $\mathcal{I}$ that model $\mathcal{S}$, we have $\mathcal{I} \models S$
- $\mathcal{S}$ is Complete if $\forall$ sentences S, we have $\mathcal{S} \models S$ or $\mathcal{S} \not\models S$; otherwise $\mathcal{S}$ is Incomplete

Question

Is the following correct? “if PA is ω-consistent, then it is Incomplete”

Lemma

suppose:

1. *$\mathcal{S}$ is consistent*
2. *all true Σ_0-sentences are provable*

then all provable Σ_0-sentences are true; in particular this holds for PA

Proof.

by definition of consistency ∎

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suppose:

1. $\mathcal{S}$ *is consistent*
2. *all true* Σ_0*-sentences are provable*

then all provable Σ_0*-sentences are true; in particular this holds for* PA

Proof.

by definition of consistency ∎

Lemma

suppose:

1. $\mathcal{S}$ *is consistent and all true* Σ_0*-sentences are provable*
2. $\mathcal{S}$ *contains a* Σ_0*-formula* $F(v_1, v_2)$ *that enumerates* P^*
3. *let* $G := \forall v_2 \neg F(\overline{f}, v_2)$, *where* $f := \ulcorner \forall v_2 \neg F(v_1, v_2) \urcorner$

then Gödel's sentence G *is true*

ω-Incompleteness

Corollary

if $\mathcal{S}$ is consistent, then Gödel's sentence G is not provable in $\mathcal{S}$, but true

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a system $\mathcal{S}$ is ω-incomplete, if $\exists$ formula $F(v_1)$, such that

$$\mathcal{S} \vdash F(\overline{0}), \ldots \mathcal{S} \vdash F(\overline{n}), \ldots$$

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Theorem

if $\mathcal{S}$ is a consistent axiomatisable system in which all Σ_0-sentences are provable, then $\mathcal{S}$ is ω-incomplete