Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

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Homework

- Chapter VI, Exercise 3
- Chapter VI, Exercise 4

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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Theorem (Abstract Gödel's Theorem)

suppose $\mathcal{L}$ is correct and $P := \{g(S) \mid S \in \mathcal{P}\}$ such that $(\sim P)^*$ is expressible in $\mathcal{L}$; then $\exists$ a true sentence of $\mathcal{L}$, not provable in $\mathcal{L}$

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Definition (PA)

...

$$N_{12}: \qquad F[\overline{0}] \rightarrow (\forall v_1(F(v_1) \rightarrow F[v_1']) \rightarrow \forall v_1 F(v_1))$$

...

Theorem

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Theorem (Gödel's First Incompleteness Theorem)

Let $\mathcal{S}$ be an extension of PA. *If $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete*

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how to fill the gap between ω-consistency and consistency

Recall

$$\Omega_4: \quad v_1 \leqslant \overline{n} \leftrightarrow (v_1 = \overline{0} \vee \cdots \vee v_1 = \overline{n})$$

$$\Omega_5: \quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1$$

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every consistent extension of Ω_4, Ω_5 which is Σ_0-complete must be incomplete

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Corollary

every consistent extension of R *is incomplete; hence* PA *is incomplete*

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Recall: A Dual to Gödel's Incompleteness

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if some superset A of R^* disjoint from P^* is representable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete; more precisely if $H(v_1)$ represents A, then the sentence $H(\overline{h})$ is undecidable, where $h := \ulcorner H(v_1) \urcorner$

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on the whiteboard

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NB: consistency is not necessary as assumption as implied by disjointness of R^* from P^*

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a formula $F(v_1)$ separates a set A from a set B in system $\mathcal{S}$, if

- $\forall\ n \in A$: $F(\overline{n})$ is provable in $\mathcal{S}$
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- hence A' and B are disjoint

∎

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Definition (General Separation)

a formula $F(v_1, \ldots, v_n)$ separates a relation $R_1(x_1, \ldots, x_n)$ from $R_2(x_1, \ldots, x_n)$ in $\mathcal{S}$, if

- $\forall\ k_1, \ldots k_n \in \mathbb{N}$ such that $R_1(k_1, \ldots, k_n)$ holds: $F(\overline{k_1}, \ldots, \overline{k_n})$ is provable in $\mathcal{S}$
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Definition

we say R_1 is separable from R_2 if some formula exists that separates R_1 from R_2

Rosser Systems

Definition (Rosser System)

a system $\mathcal{S}$ is called Rosser if

1. for any Σ_1-sets A and B, the set $A \setminus B$ is separable from $B \setminus A$
2. for any Σ_1-relations $R_1(x_1, \ldots, x_n)$ and $R_2(x_1, \ldots, x_n)$ with $n > 1$, the relation

$$R_1(x_1, \ldots, x_n) \land \neg R_2(x_1, \ldots, x_n) \qquad (R_1 \setminus R_2)$$

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Observation

suppose A and B are disjoint Σ_1-sets and $\mathcal{S}$ is a Rosser system, then A and B are separable in $\mathcal{S}$

Lemma (Separation Lemma)

if all formulas in Ω_4, Ω_5 are provable in $\mathcal{S}$, then for any two relation

$$R_1(x_1, \dots, x_n) \qquad R_2(x_1, \dots, x_n)$$

that are enumerable in $\mathcal{S}$, $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are separable

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Theorem

every extension $\mathcal{S}$ of Ω_4, Ω_5 which is Σ_0-complete is a Rosser system

Proof.

if $\mathcal{S}$ is Σ_0-complete, then all Σ_1-relations are enumerable in $\mathcal{S}$ ∎