# Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

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## Homework

- Chapter VI, Exercise 3
- Chapter VI, Exercise 4





## Outline of the Lecture







## Previous Results

**Definition**

let $g$ be a bijection such that

$$g(E) := \text{Gödel number of expression } E$$

**Theorem (Abstract Gödel's Theorem)**

*suppose $\mathcal{L}$ is correct and $P := \{g(S) \mid S \in \mathcal{P}\}$ such that $(\sim P)^*$ is expressible in $\mathcal{L}$; then $\exists$ a true sentence of $\mathcal{L}$, not provable in $\mathcal{L}$*

**Definition (PA)**

...

$$N_{12}: \quad F[\overline{0}] \to (\forall v_1(F(v_1) \to F[v_1']) \to \forall v_1 F(v_1))$$

...

### Theorem
*if PA is correct, then PA is incomplete*

### Theorem
*if PA is $\omega$-consistent, then it is incomplete*

### Theorem (Gödel's First Incompleteness Theorem)
*Let $\mathcal{S}$ be an extension of PA. If $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete*

how to fill the gap between $\omega$-consistency and consistency





### Recall
$$\Omega_4: \quad v_1 \leqslant \overline{n} \leftrightarrow (v_1 = \overline{0} \vee \cdots \vee v_1 = \overline{n})$$
$$\Omega_5: \quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1$$

### Theorem (Rosser's Theorem)
*every consistent extension of $\Omega_4, \Omega_5$ in which all $\Sigma_1$-sets are enumerable must be incomplete*

### Corollary
*every consistent extension of $\Omega_4, \Omega_5$ which is $\Sigma_0$-complete must be incomplete*

### Corollary
*every consistent extension of $R$ is incomplete; hence PA is incomplete*





## Abstract Incompleteness After Rosser

### Recall: A Dual to Gödel's Incompleteness
if $R^*$ is representable in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete

### Theorem
*if some superset $A$ of $R^*$ disjoint from $P^*$ is representable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete; more precisely if $H(v_1)$ represents $A$, then the sentence $H(\overline{h})$ is undecidable, where $h := \ulcorner H(v_1) \urcorner$*

### Proof.
on the whiteboard ∎

NB: consistency is not necessary as assumption as implied by disjointness of $R^*$ from $P^*$





### Definition
a formula $F(v_1)$ separates a set $A$ from a set $B$ in system $\mathcal{S}$, if
- $\forall\ n \in A$: $F(\overline{n})$ is provable in $\mathcal{S}$
- $\forall\ n \in B$: $F(\overline{n})$ is refutable in $\mathcal{S}$

### Lemma
*suppose $F(v_1)$ separates $A$ from $B$ in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $F(v_1)$ represents some superset of $A$ disjoint from $B$*

### Proof.
- let $A'$ be the set represented by $F(v_1)$
- since $\forall\ n \in A$: $F(\overline{n})$ is provable in $S$, we have $A \subseteq A'$
- suppose $n \in A' \cap B$; then $F(\overline{n})$ is provable and refutable
- hence $A'$ and $B$ are disjoint ∎

### Theorem

*if $H(v_1)$ separates $R^*$ from $P^*$ in $\mathcal{S}$, then the sentence $H(\overline{h})$ is undecidable in $\mathcal{S}$, where $h := \ulcorner H(v_1) \urcorner$*

### Definition (General Separation)

a formula $F(v_1, \dots, v_n)$ separates a relation $R_1(x_1, \dots, x_n)$ from $R_2(x_1, \dots, x_n)$ in $\mathcal{S}$, if

- $\forall\ k_1, \dots k_n \in \mathbb{N}$ such that $R_1(k_1, \dots, k_n)$ holds: $F(\overline{k_1}, \dots, \overline{k_n})$ is provable in $\mathcal{S}$
- $\forall\ k_1, \dots k_n \in \mathbb{N}$ such that $R_1(k_1, \dots, k_n)$ holds: $F(\overline{k_1}, \dots, \overline{k_n})$ is refutable in $\mathcal{S}$

### Definition

we say $R_1$ is separable from $R_2$ if some formula exists that separates $R_1$ from $R_2$





## Rosser Systems

### Definition (Rosser System)

a system $\mathcal{S}$ is called Rosser if

1. for any $\Sigma_1$-sets $A$ and $B$, the set $A \setminus B$ is separable from $B \setminus A$
2. for any $\Sigma_1$-relations $R_1(x_1, \dots, x_n)$ and $R_2(x_1, \dots, x_n)$ with $n > 1$, the relation

$$R_1(x_1, \dots, x_n) \land \neg R_2(x_1, \dots, x_n) \qquad (R_1 \setminus R_2)$$

is separable from

$$R_2(x_1, \dots, x_n) \land \neg R_1(x_1, \dots, x_n) \qquad (R_2 \setminus R_1)$$

### Observation

suppose $A$ and $B$ are disjoint $\Sigma_1$-sets and $\mathcal{S}$ is a Rosser system, then $A$ and $B$ are separable in $\mathcal{S}$





### Lemma (Separation Lemma)

*if all formulas in $\Omega_4$, $\Omega_5$ are provable in $\mathcal{S}$, then for any two relation*

$$R_1(x_1, \dots, x_n) \qquad R_2(x_1, \dots, x_n)$$

*that are enumerable in $\mathcal{S}$, $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are separable*

### Proof.

on the whiteboard ∎

### Theorem

*every extension $\mathcal{S}$ of $\Omega_4, \Omega_5$ which is $\Sigma_0$-complete is a Rosser system*

### Proof.

if $\mathcal{S}$ is $\Sigma_0$-complete, then all $\Sigma_1$-relations are enumerable in $\mathcal{S}$ ∎