

Homework

- Chapter VI, Exercise 3
- Chapter VI, Exercise 4

Gödel's Incompleteness Theorem

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Outline

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of \mathcal{L}

Tarski's Theorem for Arithmetic

the language \mathcal{L}_E , concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1 -relations

Gödel's Proof

ω -consistency, a basic incompleteness theorem, ω -consistency lemma, Σ_0 -complete subsystems, ω -incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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Gödel's Incompleteness Theorem

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Previous Results

Previous Results

Definition

let g be a bijection such that

$$g(E) := \text{Gödel number of expression } E$$

Theorem (Abstract Gödel's Theorem)

suppose \mathcal{L} is correct and $P := \{g(S) \mid S \in \mathcal{P}\}$ such that $(\sim P)^*$ is expressible in \mathcal{L} ; then \exists a true sentence of \mathcal{L} , not provable in \mathcal{L}

Definition (PA)

...

$$N_{12}: \quad F[\bar{0}] \rightarrow (\forall v_1 (F(v_1) \rightarrow F[v'_1]) \rightarrow \forall v_1 F(v_1))$$

...

Theorem

if PA is correct, then PA is incomplete

Theorem

if PA is ω -consistent, then it is incomplete

Theorem (Gödel's First Incompleteness Theorem)

Let S be an extension of PA. If S is consistent, then S is incomplete

how to fill the gap between ω -consistency and consistency

Recall

$$\Omega_4 : \quad v_1 \leq \bar{n} \leftrightarrow (v_1 = \bar{0} \vee \dots \vee v_1 = \bar{n})$$

$$\Omega_5 : \quad v_1 \leq \bar{n} \vee \bar{n} \leq v_1$$

Theorem (Rosser's Theorem)

every consistent extension of Ω_4, Ω_5 in which all Σ_1 -sets are enumerable must be incomplete

Corollary

every consistent extension of Ω_4, Ω_5 which is Σ_0 -complete must be incomplete

Corollary

every consistent extension of R is incomplete; hence PA is incomplete

Abstract Incompleteness After Rosser

Recall: A Dual to Gödel's Incompleteness

if R^* is representable in S and S is consistent, then S is incomplete

Theorem

if some superset A of R^* disjoint from P^* is representable in S , then S is incomplete; more precisely if $H(v_1)$ represents A , then the sentence $H(\bar{h})$ is undecidable, where $h := \ulcorner H(v_1) \urcorner$

Proof.

on the whiteboard ■

NB: consistency is not necessary as assumption as implied by disjointness of R^* from P^*

Definition

a formula $F(v_1)$ separates a set A from a set B in system S , if

- $\forall n \in A: F(\bar{n})$ is provable in S
- $\forall n \in B: F(\bar{n})$ is refutable in S

Lemma

suppose $F(v_1)$ separates A from B in S and S is consistent, then $F(v_1)$ represents some superset of A disjoint from B

Proof.

- let A' be the set represented by $F(v_1)$
- since $\forall n \in A: F(\bar{n})$ is provable in S , we have $A \subseteq A'$
- suppose $n \in A' \cap B$; then $F(\bar{n})$ is provable and refutable
- hence A' and B are disjoint ■

Theorem

if $H(v_1)$ separates R^* from P^* in \mathcal{S} , then the sentence $H(\bar{h})$ is undecidable in \mathcal{S} , where $h := \ulcorner H(v_1) \urcorner$

Definition (General Separation)

a formula $F(v_1, \dots, v_n)$ **separates** a relation $R_1(x_1, \dots, x_n)$ from $R_2(x_1, \dots, x_n)$ in \mathcal{S} , if

- $\forall k_1, \dots, k_n \in \mathbb{N}$ such that $R_1(k_1, \dots, k_n)$ holds: $F(\bar{k}_1, \dots, \bar{k}_n)$ is provable in \mathcal{S}
- $\forall k_1, \dots, k_n \in \mathbb{N}$ such that $R_2(k_1, \dots, k_n)$ holds: $F(\bar{k}_1, \dots, \bar{k}_n)$ is refutable in \mathcal{S}

Definition

we say R_1 is **separable** from R_2 if some formula exists that separates R_1 from R_2

Lemma (Separation Lemma)

if all formulas in Ω_4, Ω_5 are provable in \mathcal{S} , then for any two relation

$$R_1(x_1, \dots, x_n) \quad R_2(x_1, \dots, x_n)$$

that are **enumerable** in \mathcal{S} , $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are separable

Proof.

on the whiteboard ■

Theorem

every extension \mathcal{S} of Ω_4, Ω_5 which is Σ_0 -complete is a Rosser system

Proof.

if \mathcal{S} is Σ_0 -complete, then all Σ_1 -relations are enumerable in \mathcal{S} ■

Rosser Systems

Definition (Rosser System)

a system \mathcal{S} is called **Rosser** if

- 1 for any Σ_1 -sets A and B , the set $A \setminus B$ is separable from $B \setminus A$
- 2 for any Σ_1 -relations $R_1(x_1, \dots, x_n)$ and $R_2(x_1, \dots, x_n)$ with $n > 1$, the relation

$$R_1(x_1, \dots, x_n) \wedge \neg R_2(x_1, \dots, x_n) \quad (R_1 \setminus R_2)$$

is separable from

$$R_2(x_1, \dots, x_n) \wedge \neg R_1(x_1, \dots, x_n) \quad (R_2 \setminus R_1)$$

Observation

suppose A and B are disjoint Σ_1 -sets and \mathcal{S} is a Rosser system, then A and B are separable in \mathcal{S}