# Gödel's Incompleteness Theorem

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# Summary of Last Lecture

## Lemma (Separation Lemma)

*if all formulas in $\Omega_4$, $\Omega_5$ are provable in $\mathcal{S}$, then for any two relation*

$$R_1(x_1, \dots, x_n) \qquad R_2(x_1, \dots, x_n)$$

*that are enumerable in $\mathcal{S}$, $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are separable*

## Theorem

*every extension $\mathcal{S}$ of $\Omega_4, \Omega_5$ which is $\Sigma_0$-complete is a Rosser system*

## Corollary

*the systems $R$, $Q$, and* PA *are Rosser systems*

# Homework

- Chapter VI, Exercise 6.

# Outline of the Lecture

## General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

## Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, incompleteness of PA, $\Sigma_1$-relations

## Gödel's Proof

$\omega$-consistency, $\Sigma_0$-complete subsystems, $\omega$-incompleteness of PA

## Rosser Systems

general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared

## The Unprovability of Consistency

definability and diagonalisation, the unprovability of consistency

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# Rosser's Undecidable Sentence

## Theorem

*if $\mathcal{S}$ is a consistent system that extends $\Omega_4, \Omega_5$ such that $P^*$ and $R^*$ are enumerable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete*

# Rosser's Undecidable Sentence

**Theorem**

*if $\mathcal{S}$ is a consistent system that extends $\Omega_4, \Omega_5$ such that $P^*$ and $R^*$ are enumerable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete*

**Proof.**

on the whiteboard

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*every consistent extension of $\Omega_4, \Omega_5$ in which all $\Sigma_1$-sets are enumerable must be incomplete*

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# Gödel and Rosser Sentences Compared

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- Rosser's sentence $\forall y (A(\overline{h}, y) \rightarrow (\exists z \leqslant y) B(\overline{h}, z))$ expresses that for any potential witness of provability, there exists a potential smaller witness of refutability

# Definability and Complete Representability

## Recall

$F(v_1, \ldots, v_n)$ represents $R$ in $\mathcal{S}$ if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \ldots, \overline{m}_n) \text{ is provable in } \mathcal{S} \iff (m_1, \ldots, m_n) \in R$$

we also say that $F(v_1, \ldots, v_n)$ represents the relation $R(x_1, \ldots, x_n)$

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1. if $R(m_1, \dots, m_n)$ holds, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is provable in $\mathcal{S}$
2. if $R(m_1, \dots, m_n)$ is false, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is refutable in $\mathcal{S}$

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$F(v_1, \dots, v_n)$ completely represents $R$ in $\mathcal{S}$ if

1. $F$ represents $R$
2. $\neg F$ represents $\sim R$

## Lemma

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2. *If $\mathcal{S}$ is a consistent Rosser system, then all recursive relations are completely representable in $\mathcal{S}$*

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## Proof.

- by definition $R \in \Sigma_1$ and $\sim R \in \Sigma_1$ and by assumption $\exists$ formula $F(v_1)$ that separates $R$ from $\sim R$
- hence $F$ defines $R$ ∎

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  If $f(a_1, \dots, a_n) = b$, then
  $\forall v_{n+1} F(\overline{a}_1, \dots, \overline{a}_n, v_{n+1}) \to v_{n+1} = \overline{b}$ is provable in $\mathcal{S}$

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### Theorem

*if $f(x)$ is strongly definable in $\mathcal{S}$, then for any formula $G(v_1)$, there $\exists$ a formula $H(v_1)$ such that $\forall\ n \in \mathbb{N}$, $H(\overline{n}) \leftrightarrow G(\overline{f(n)})$ is provable*

### Definition

for any disjoint pair $(A, B)$ of sets, a formula $F(v_1)$ exactly separates $A$ from $B$ in $\mathcal{S}$, if $F(v_1)$ represents $A$ and $\neg F(v_1)$ represents $B$

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## Corollary

*suppose $f(x)$ is strongly definable in $\mathcal{S}$*

1. $\forall$ *sets $A$ representable in $\mathcal{S}$, $f^{-1}(A)$ is representable in $\mathcal{S}$*
2. $\forall$ *pairs $(A, B)$ that is exactly separable in $\mathcal{S}$, the pair $(f^{-1}(A), f^{-1}(B))$ is exactly separable in $\mathcal{S}$*
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## Theorem

*all recursive functions are strongly definable in Robinson's $R$ (and hence also in* PA*)*

## Corollary

*the diagonal function $d(x)$ is strongly definable in every extension of $R$*

## Proof.

recall that any function whose graph is $\Sigma_1$, is recursive ∎