Gödel's Incompleteness Theorem

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Summary of Last Lecture

Lemma (Separation Lemma)

if all formulas in Ω_4, Ω_5 are provable in $\mathcal{S}$, then for any two relation

$$R_1(x_1, \dots, x_n) \qquad R_2(x_1, \dots, x_n)$$

that are enumerable in $\mathcal{S}$, $R_1 \setminus R_2$ and $R_2 \setminus R_1$ are separable

Theorem

every extension $\mathcal{S}$ of Ω_4, Ω_5 which is Σ_0-complete is a Rosser system

Corollary

the systems R, Q, and PA *are Rosser systems*

Homework

- Chapter VI, Exercise 6.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared

The Unprovability of Consistency

definability and diagonalisation, the unprovability of consistency

Rosser's Undecidable Sentence

Theorem

if $\mathcal{S}$ is a consistent system that extends Ω_4, Ω_5 such that P^* and R^* are enumerable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete

Proof.

on the whiteboard ∎

Corollary (Rosser's Theorem)

every consistent extension of Ω_4, Ω_5 in which all Σ_1-sets are enumerable must be incomplete

Proof.

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Gödel and Rosser Sentences Compared

consider PA

Definition

let $\exists y A(x, y)$ represent P^* and let $\exists y B(x, y)$ represent R^*

- if $A(\overline{n}, \overline{m})$ is true then we say m is a witness that $E_n(\overline{n})$ is provable
- if $B(\overline{n}, \overline{m})$ is true then we say m is a witness that $E_n(\overline{n})$ is refutable

- Gödel's sentence $\forall y \neg A(\overline{a}, y)$ expresses that for all y, y is not a witness that $E_a(\overline{a})$ is provable. Or simpler: Gödel's sentence expresses its own unprovability
- Rosser's sentence $\forall y (A(\overline{h}, y) \rightarrow (\exists z \leqslant y) B(\overline{h}, z))$ expresses that for any potential witness of provability, there exists a potential smaller witness of refutability

Definability and Complete Representability

Recall

$F(v_1, \dots, v_n)$ represents R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is provable in } \mathcal{S} \iff (m_1, \dots, m_n) \in R$$

we also say that $F(v_1, \dots, v_n)$ represents the relation $R(x_1, \dots, x_n)$

Definition

$F(v_1, \dots, v_n)$ defines R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

1. if $R(m_1, \dots, m_n)$ holds, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is provable in $\mathcal{S}$
2. if $R(m_1, \dots, m_n)$ is false, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is refutable in $\mathcal{S}$

$F(v_1, \dots, v_n)$ completely represents R in $\mathcal{S}$ if

1. F represents R
2. $\neg F$ represents $\sim R$

Lemma

If F defines R and $\mathcal{S}$ is consistent then F completely represents R in $\mathcal{S}$

Proof.

on the whiteboard ∎

Lemma

1. *If $\mathcal{S}$ is a Rosser system, then all recursive relations are definable in $\mathcal{S}$*
2. *If $\mathcal{S}$ is a consistent Rosser system, then all recursive relations are completely representable in $\mathcal{S}$*

Proof.

- by definition $R \in \Sigma_1$ and $\sim R \in \Sigma_1$ and by assumption $\exists$ formula $F(v_1)$ that separates R from $\sim R$
- hence F defines R ∎

Theorem

all recursive relations are definable in Robinson's R (and also in PA)

Definition

- $F(v_1, \dots, v_n, v_{n+1})$ weakly defines the function $f(x_1, \dots, x_n)$ in $\mathcal{S}$ if F defines the following relation:

$$f(x_1, \dots, x_n) = x_{n+1}$$

- $F(v_1, \dots, v_n, v_{n+1})$ strongly defines the function $f(x_1, \dots, x_n)$ in $\mathcal{S}$ if F weakly defines f and the following condition holds:

  If $f(a_1, \dots, a_n) = b$, then
  $\forall v_{n+1} F(\overline{a}_1, \dots, \overline{a}_n, v_{n+1}) \rightarrow v_{n+1} = \overline{b}$ is provable in $\mathcal{S}$

Theorem

if $f(x)$ is strongly definable in $\mathcal{S}$, then for any formula $G(v_1)$, there $\exists$ a formula $H(v_1)$ such that $\forall\ n \in \mathbb{N}$, $H(\overline{n}) \leftrightarrow G(\overline{f(n)})$ is provable

Definition

for any disjoint pair (A, B) of sets, a formula $F(v_1)$ exactly separates A from B in $\mathcal{S}$, if $F(v_1)$ represents A and $\neg F(v_1)$ represents B

Corollary

suppose $f(x)$ is strongly definable in $\mathcal{S}$

1. *$\forall$ sets A representable in $\mathcal{S}$, $f^{-1}(A)$ is representable in $\mathcal{S}$*
2. *$\forall$ pairs (A, B) that is exactly separable in $\mathcal{S}$, the pair $(f^{-1}(A), f^{-1}(B))$ is exactly separable in $\mathcal{S}$*
3. *$\forall$ sets A definable in $\mathcal{S}$, $f^{-1}(A)$ is definable in $\mathcal{S}$*

Proof.

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Lemma

if $\mathcal{S}$ is an extension of Ω_4, Ω_5, then any function f weakly definable in $\mathcal{S}$, is strongly definable in $\mathcal{S}$

Proof.

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Theorem

all recursive functions are strongly definable in Robinson's R (and hence also in PA)

Corollary

the diagonal function $d(x)$ is strongly definable in every extension of R

Proof.

recall that any function whose graph is Σ_1, is recursive ■