Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

Winter 2011

Homework

- Property 3) in Lemma on page 101
- Chapter VIII, Exercise 4.
- Chapter VIII, Exercise 5.
- Chapter VIII, Exercise 6.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared

The Unprovability of Consistency

definability and diagonalisation, the unprovability of consistency

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared

The Unprovability of Consistency

definability and diagonalisation, the unprovability of consistency

Definability and Complete Representability Revisited

Definition

- $F(v_1, \dots, v_n)$ represents R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is provable in } \mathcal{S} \iff (m_1, \dots, m_n) \in R$$

- $F(v_1, \dots, v_n)$ defines R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:
  1. if $R(m_1, \dots, m_n)$ holds, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is provable in $\mathcal{S}$
  2. if $R(m_1, \dots, m_n)$ is false, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is refutable in $\mathcal{S}$
- $F(v_1, \dots, v_n)$ completely represents R in $\mathcal{S}$ if
  1. F represents R
  2. $\neg F$ represents $\sim R$

Definability and Complete Representability Revisited

Definition

- $F(v_1, \ldots, v_n)$ represents R in $\mathcal{S}$ if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \ldots, \overline{m}_n) \text{ is provable in } \mathcal{S} \iff (m_1, \ldots, m_n) \in R$$

- $F(v_1, \ldots, v_n)$ defines R in $\mathcal{S}$ if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:
  1. if $R(m_1, \ldots, m_n)$ holds, then $F(\overline{m}_1, \ldots, \overline{m}_n)$ is provable in $\mathcal{S}$
  2. if $R(m_1, \ldots, m_n)$ is false, then $F(\overline{m}_1, \ldots, \overline{m}_n)$ is refutable in $\mathcal{S}$
- $F(v_1, \ldots, v_n)$ completely represents R in $\mathcal{S}$ if
  1. F represents R
  2. $\neg F$ represents $\sim R$

Remark

if $\mathcal{S}$ is consistent:

completely representability $\Leftrightarrow$ definability $\Rightarrow$ representability

for any expression X, $\overline{X} := \overline{\ulcorner X \urcorner}$

Definition

a sentence X is called fixed point of formula $F(v)$ (in $\mathcal{S}$) if
$\mathcal{S} \vdash F(\overline{X}) \leftrightarrow X$

for any expression X, $\overline{X} := \overline{\ulcorner X \urcorner}$

Definition

a sentence X is called fixed point of formula $F(v)$ (in $\mathcal{S}$) if
$\mathcal{S} \vdash F(\overline{X}) \leftrightarrow X$

Theorem

if the diagonal function $d(x)$ is strongly definable in $\mathcal{S}$, then every formula has a fixed point

Proof.

on the whiteboard

for any expression X, $\overline{X} := \overline{\ulcorner X \urcorner}$

Definition

a sentence X is called fixed point of formula $F(v)$ (in $\mathcal{S}$) if
$\mathcal{S} \vdash F(\overline{X}) \leftrightarrow X$

Theorem

if the diagonal function $d(x)$ is strongly definable in $\mathcal{S}$, then every formula has a fixed point

Proof.

on the whiteboard ∎

for any expression X, $\overline{X} := \overline{\ulcorner X \urcorner}$

Definition

a sentence X is called fixed point of formula $F(v)$ (in $\mathcal{S}$) if
$\mathcal{S} \vdash F(\overline{X}) \leftrightarrow X$

Theorem

if the diagonal function $d(x)$ is strongly definable in $\mathcal{S}$, then every formula has a fixed point

Proof.

on the whiteboard ∎

Corollary

if $\mathcal{S}$ is an extension of Robinson's R, then every formula has a fixed point

Gödel Sentences and Fixed Points

Definition

we say X is a Gödel sentence for set A with respect to $\mathcal{S}$ if X is provable in $\mathcal{S}$ iff A contains $\ulcorner X \urcorner$

Gödel Sentences and Fixed Points

Definition

we say X is a Gödel sentence for set A with respect to $\mathcal{S}$ if X is provable in $\mathcal{S}$ iff A contains $\ulcorner X \urcorner$

Definition

function $f(x)$ is called acceptable in $\mathcal{S}$ if for every set A representable in $\mathcal{S}$, $f^{-1}(A)$ is representable in $\mathcal{S}$

Gödel Sentences and Fixed Points

Definition

we say X is a Gödel sentence for set A with respect to $\mathcal{S}$ if X is provable in $\mathcal{S}$ iff A contains $\ulcorner X \urcorner$

Definition

function $f(x)$ is called acceptable in $\mathcal{S}$ if for every set A representable in $\mathcal{S}$, $f^{-1}(A)$ is representable in $\mathcal{S}$

Theorem

if the diagonal function $d(x)$ is acceptable in $\mathcal{S}$, then for every set A representable in $\mathcal{S}$, there is a Gödel sentence for A

NB: if X is a Gödel sentence for a set represented by $F(v_1)$, then X is provable iff $F(\overline{X})$ is provable

Truth Predicates

Definition

a formula $T(v_1)$ is a truth-predicate for $\mathcal{S}$ if for every sentence X:
$\mathcal{S} \vdash X \leftrightarrow T(\overline{X})$

Truth Predicates

Definition

a formula $T(v_1)$ is a truth-predicate for $\mathcal{S}$ if for every sentence X:
$\mathcal{S} \vdash X \leftrightarrow T(\overline{X})$

Theorem

if $\mathcal{S}$ is correct, then there is no truth predicate for $\mathcal{S}$

Truth Predicates

Definition

a formula $T(v_1)$ is a truth-predicate for $\mathcal{S}$ if for every sentence X:
$\mathcal{S} \vdash X \leftrightarrow T(\overline{X})$

Theorem

if $\mathcal{S}$ is correct, then there is no truth predicate for $\mathcal{S}$

Theorem

if $\mathcal{S}$ is consistent and the diagonal function is strongly definable, then there is not truth-predicate for $\mathcal{S}$

Definition

a formula $P(v_1)$ is called provability predicate for $\mathcal{S}$ if

- P_1 if X is provable in $\mathcal{S}$, then so is $P(\overline{X})$
- P_2 $P(\overline{X \rightarrow Y}) \rightarrow (P(\overline{X}) \rightarrow P(\overline{Y}))$ is provable in $\mathcal{S}$
- P_3 $P(\overline{X}) \rightarrow P(P(\overline{X}))$ is provable in $\mathcal{S}$

Definition

a formula $P(v_1)$ is called provability predicate for $\mathcal{S}$ if

- P_1 if X is provable in $\mathcal{S}$, then so is $P(\overline{X})$
- P_2 $P(\overline{X \to Y}) \to (P(\overline{X}) \to P(\overline{Y}))$ is provable in $\mathcal{S}$
- P_3 $P(\overline{X}) \to P(P(\overline{X}))$ is provable in $\mathcal{S}$

Lemma

let $P(v_1)$ be a provability predicate, then we obtain:

- P_4 *if $X \to Y$ is provable, so is $P(\overline{X}) \to P(\overline{Y})$*
- P_5 *if $X \to (Y \to Z)$ is provable, so is $P(\overline{X}) \to (P(\overline{Y}) \to P(\overline{Z}))$*
- P_6 *if $X \to (P(\overline{X}) \to Y)$ is provable, so is $P(\overline{X}) \to P(\overline{Y})$*

Proof.

on the whiteboard

Definition

a formula $P(v_1)$ is called provability predicate for $\mathcal{S}$ if

- P_1 if X is provable in $\mathcal{S}$, then so is $P(\overline{X})$
- P_2 $P(\overline{X \to Y}) \to (P(\overline{X}) \to P(\overline{Y}))$ is provable in $\mathcal{S}$
- P_3 $P(\overline{X}) \to P(P(\overline{X}))$ is provable in $\mathcal{S}$

Lemma

let $P(v_1)$ be a provability predicate, then we obtain:

- P_4 *if $X \to Y$ is provable, so is $P(\overline{X}) \to P(\overline{Y})$*
- P_5 *if $X \to (Y \to Z)$ is provable, so is $P(\overline{X}) \to (P(\overline{Y}) \to P(\overline{Z}))$*
- P_6 *if $X \to (P(\overline{X}) \to Y)$ is provable, so is $P(\overline{X}) \to P(\overline{Y})$*

Proof.

on the whiteboard ∎

Definition

$\mathcal{S}$ is called diagonalisable if every formula has a fixed point

Definition

$\mathcal{S}$ is called diagonalisable if every formula has a fixed point

Example

PA is diagonalisable

Definition

$\mathcal{S}$ is called diagonalisable if every formula has a fixed point

Example

PA is diagonalisable

Convention

in the following $P(v_1)$ denotes a provability predicate

Definition

let consis $:= \neg P(\overline{\overline{0} = \overline{1}})$

Abstract Form of Gödel's Second Incompleteness Theorem

Theorem

if G is a fixed point of the formula $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$

Proof.

on the whiteboard

Abstract Form of Gödel's Second Incompleteness Theorem

Theorem

if G is a fixed point of the formula $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$

Proof.

on the whiteboard ∎

Abstract Form of Gödel's Second Incompleteness Theorem

Theorem

if G is a fixed point of the formula $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$

Proof.

on the whiteboard ∎

Theorem

if G is a fixed point of $\neg P(v_1)$, then $\text{consis} \rightarrow G$ *is provable in $\mathcal{S}$*

Proof.

on the whiteboard

Abstract Form of Gödel's Second Incompleteness Theorem

Theorem

if G is a fixed point of the formula $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$

Proof.

on the whiteboard ∎

Theorem

if G is a fixed point of $\neg P(v_1)$, then consis $\rightarrow G$ *is provable in $\mathcal{S}$*

Proof.

on the whiteboard ∎

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard ∎

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard ∎

Theorem (Löb's Theorem)

suppose $\mathcal{S}$ is diagonalisable; then for any sentence Y, if $P(\overline{Y}) \to Y$ is provable in S, so is Y

Proof.

on the whiteboard

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard ∎

Theorem (Löb's Theorem)

suppose $\mathcal{S}$ is diagonalisable; then for any sentence Y, if $P(\overline{Y}) \rightarrow Y$ is provable in S, so is Y

Proof.

on the whiteboard ∎

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard ∎

Theorem (Löb's Theorem)

suppose $\mathcal{S}$ is diagonalisable; then for any sentence Y, if $P(\overline{Y}) \to Y$ is provable in S, so is Y

Proof.

on the whiteboard ∎

NB: Gödel's Second Incompleteness Theorem is a corollary to Löb's Theorem

To Conclude

Discussion on FOM

...

- WV: *Consistency on the other hand is not an interesting problem since it has been shown by Gödel to be impossible to proof.*
- HF: *How would you assess the various proofs of the consistency of PA that we now have, for your purposes?* [...]
- WV: *Well, that would contradict Gödel's result.*
- ...

To Conclude

Discussion on FOM

...

- WV: *Consistency on the other hand is not an interesting problem since it has been shown by Gödel to be impossible to proof.*
- HF: *How would you assess the various proofs of the consistency of PA that we now have, for your purposes?* [...]
- WV: *Well, that would contradict Gödel's result.*
- ...

Smullyan says: Rubbish!

Thank You for Your Attention!