Computational Logic

Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

Winter 2011

Homework

- Property 3) in Lemma on page 101
- Chapter VIII, Exercise 4.
- Chapter VIII, Exercise 5.
- Chapter VIII, Exercise 6.

Outline of the Lecture

Definability and Complete Representability Revisited

Definition

- $F(v_1, \dots, v_n)$ represents R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is provable in } \mathcal{S} \iff (m_1, \dots, m_n) \in R$$

- $F(v_1, \dots, v_n)$ defines R in $\mathcal{S}$ if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:
  1. if $R(m_1, \dots, m_n)$ holds, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is provable in $\mathcal{S}$
  2. if $R(m_1, \dots, m_n)$ is false, then $F(\overline{m}_1, \dots, \overline{m}_n)$ is refutable in $\mathcal{S}$
- $F(v_1, \dots, v_n)$ completely represents R in $\mathcal{S}$ if
  1. F represents R
  2. $\neg F$ represents $\sim R$

Remark

if $\mathcal{S}$ is consistent:

completely representability $\Leftrightarrow$ definability $\Rightarrow$ representability for any expression X, $\overline{X} := \ulcorner X \urcorner$

Definition

a sentence X is called fixed point of formula $F(v)$ (in $\mathcal{S}$) if
$\mathcal{S} \vdash F(\overline{X}) \leftrightarrow X$

Theorem

if the diagonal function $d(x)$ is strongly definable in $\mathcal{S}$, then every formula has a fixed point

Proof.

on the whiteboard ■

Corollary

if $\mathcal{S}$ is an extension of Robinson's R, then every formula has a fixed point

Gödel Sentences and Fixed Points

Definition

we say X is a Gödel sentence for set A with respect to $\mathcal{S}$ if X is provable in $\mathcal{S}$ iff A contains $\ulcorner X \urcorner$

Definition

function $f(x)$ is called acceptable in $\mathcal{S}$ if for every set A representable in $\mathcal{S}$, $f^{-1}(A)$ is representable in $\mathcal{S}$

Theorem

if the diagonal function $d(x)$ is acceptable in $\mathcal{S}$, then for every set A representable in $\mathcal{S}$, there is a Gödel sentence for A

NB: if X is a Gödel sentence for a set represented by $F(v_1)$, then X is provable iff $F(\overline{X})$ is provable

Truth Predicates

Definition

a formula $T(v_1)$ is a truth-predicate for $\mathcal{S}$ if for every sentence X:
$\mathcal{S} \vdash X \leftrightarrow T(\overline{X})$

Theorem

if $\mathcal{S}$ is correct, then there is no truth predicate for $\mathcal{S}$

Theorem

if $\mathcal{S}$ is consistent and the diagonal function is strongly definable, then there is not truth-predicate for $\mathcal{S}$

Definition

a formula $P(v_1)$ is called provability predicate for $\mathcal{S}$ if

- P_1 if X is provable in $\mathcal{S}$, then so is $P(\overline{X})$
- P_2 $P(\overline{X \rightarrow Y}) \rightarrow (P(\overline{X}) \rightarrow P(\overline{Y}))$ is provable in $\mathcal{S}$
- P_3 $P(\overline{X}) \rightarrow P(\overline{P(\overline{X})})$ is provable in $\mathcal{S}$

Lemma

let $P(v_1)$ be a provability predicate, then we obtain:

- P_4 *if $X \rightarrow Y$ is provable, so is $P(\overline{X}) \rightarrow P(\overline{Y})$*
- P_5 *if $X \rightarrow (Y \rightarrow Z)$ is provable, so is $P(\overline{X}) \rightarrow (P(\overline{Y}) \rightarrow P(\overline{Z}))$*
- P_6 *if $X \rightarrow (P(\overline{X}) \rightarrow Y)$ is provable, so is $P(\overline{X}) \rightarrow P(\overline{Y})$*

Proof.

on the whiteboard ■

Definition

$\mathcal{S}$ is called diagonalisable if every formula has a fixed point

Example

PA is diagonalisable

Convention

in the following $P(v_1)$ denotes a provability predicate

Definition

let consis $:= \neg P(\overline{\overline{0} = \overline{1}})$

Abstract Form of Gödel's Second Incompleteness Theorem

Theorem

if G is a fixed point of the formula $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$

Proof.

on the whiteboard ■

Theorem

if G is a fixed point of $\neg P(v_1)$, then consis $\rightarrow G$ *is provable in $\mathcal{S}$*

Proof.

on the whiteboard ■

Theorem

suppose $\mathcal{S}$ is diagonalisable and consistent; then consis *is not provable in $\mathcal{S}$*

Proof.

on the whiteboard ■

Theorem (Löb's Theorem)

suppose $\mathcal{S}$ is diagonalisable; then for any sentence Y, if $P(\overline{Y}) \rightarrow Y$ is provable in S, so is Y

Proof.

on the whiteboard ■

NB: Gödel's Second Incompleteness Theorem is a corollary to Löb's Theorem

To Conclude

Discussion on FOM

...

- WV: *Consistency on the other hand is not an interesting problem since it has been shown by Gödel to be impossible to proof.*
- HF: *How would you assess the various proofs of the consistency of PA that we now have, for your purposes?* [...]
- WV: *Well, that would contradict Gödel's result.*
- ...

Smullyan says: Rubbish!

Thank You for Your Attention!