Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

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Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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3. $\exists\ \mathcal{P} \subseteq \mathcal{S}$, *the provable sentences*

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5. $\exists\, \mathcal{H} \subseteq \mathcal{E}$, $\mathcal{H}$ *are the predicates of* $\mathcal{L}$, *that is* $H \in \mathcal{H}$ *names a set of natural numbers*

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5. $\exists\, \mathcal{H} \subseteq \mathcal{E}$, $\mathcal{H}$ *are the predicates of* $\mathcal{L}$, *that is* $H \in \mathcal{H}$ *names a set of natural numbers*
6. $\exists$ *function* Φ *that maps expression* E *and number* n *to* $E(n)$; *for predicates* $H(n)$ *has to be a sentence: the sentences* $H(n)$ *expresses that* n *belongs to the set named by* H

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7. $\exists\ \mathcal{T} \subseteq \mathcal{S}$, *the true sentences*

Expressibility in $\mathcal{L}$

Definition

1. (clearly) a predicate H is true for $n \in \mathbb{N}$, if $H(n)$ holds
2. H expresses the set $\{n \mid H(n)\}$, that is A is expressed by H if

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Fact

- *there are only countable many predicates of $\mathcal{L}$; however there are non-countable many sets over $\mathbb{N}$*
- *hence not every set is expressible*

Gödel Numbering and Diagonalisation

Definition

$\mathcal{L}$ is correct if $\mathcal{P} \subseteq \mathcal{T}$ and $\mathcal{R} \cap \mathcal{T} = \varnothing$

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4. define the diagonal function d as follows:

$$d(n) := g(E_n(n))$$

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Theorem (Gödel's Theorem)

suppose $\mathcal{L}$ is correct and $P := \{g(S) \mid S \in \mathcal{P}\}$ such that $(\sim P)^*$ is expressible in $\mathcal{L}$; then $\exists$ a true sentence of $\mathcal{L}$, not provable in $\mathcal{L}$

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Proof.

on black board ∎

Fact

$(\sim P)^*$ *is expressible boils down to*

- G1 $\forall$ *sets* A *expressible in* $\mathcal{L}$, A^* *is expressible in* $\mathcal{L}$
- G2 $\forall$ *sets* A *expressible in* $\mathcal{L}$, $\sim A$ *is expressible in* $\mathcal{L}$
- G3 P *is expressible in* $\mathcal{L}$

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G3 *P is expressible in $\mathcal{L}$*

it will be simply to verify G1, it will be trivial to verify G2, but the hard part will be to verify G3

Gödel Sentences

Definition

E_n is a Gödel sentence for a number set A, if

1. either $E_n \in \mathcal{T}$ and $n \in A$, or
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Lemma (Diagonal Lemma)

1. *$\forall$ sets A, if A^* is expressible in $\mathcal{L}$, then $\exists$ Gödel sentence for A*

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2. *if $\mathcal{L}$ satisfies G1, then $\forall$ sets A expressible in $\mathcal{L}$, then $\exists$ Gödel sentence for A*

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on black board ∎

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suppose $\mathcal{L}$ is correct and let $R := \{g(S) \mid S \in \mathcal{S} \cap \mathcal{R}\}$

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2. *moreover if K expresses R^*, then its diagonalisation $K(k)$ is undecidable*

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