Gödel's Incompleteness Theorem

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Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Abstract Forms of Gödel's and Tarkis's Theorems

Fact

Gödel's argument is applicable to $\mathcal{L}$ if at least the following holds:

1. *$\exists$ countable set of expressions $\mathcal{E}$*
2. *$\exists\, \mathcal{S} \subseteq \mathcal{E}$, $\mathcal{S}$ are the sentences*
3. *$\exists\, \mathcal{P} \subseteq \mathcal{S}$, the provable sentences*
4. *$\exists\, \mathcal{R} \subseteq \mathcal{S}$, the refutable sentences*
5. *$\exists\, \mathcal{H} \subseteq \mathcal{E}$, $\mathcal{H}$ are the predicates of $\mathcal{L}$, that is $H \in \mathcal{H}$ names a set of natural numbers*
6. *$\exists$ function Φ that maps expression E and number n to $E(n)$; for predicates $H(n)$ has to be a sentence: the sentences $H(n)$ expresses that n belongs to the set named by H*
7. *$\exists\, \mathcal{T} \subseteq \mathcal{S}$, the true sentences*

Expressibility in $\mathcal{L}$

Definition

1. (clearly) a predicate H is true for $n \in \mathbb{N}$, if $H(n)$ holds
2. H expresses the set $\{n \mid H(n)\}$, that is A is expressed by H if

$$H(n) \in \mathcal{T} \iff n \in A$$

Definition

a set A is expressible (nameable) in $\mathcal{L}$ if A is expressed by some predicate

Fact

- *there are only countable many predicates of $\mathcal{L}$; however there are non-countable many sets over $\mathbb{N}$*
- *hence not every set is expressible*

Gödel Numbering and Diagonalisation

Definition
$\mathcal{L}$ is correct if $\mathcal{P} \subseteq \mathcal{T}$ and $\mathcal{R} \cap \mathcal{T} = \varnothing$

Definition
1. let g be a bijection such that

$$g(E) := \text{Gödel number of expression } E$$

2. E_n is defined such that $g(E_n) = n$
3. the diagonalisation of E_n is $E_n(n)$
4. define the diagonal function d as follows:

$$d(n) := g(E_n(n))$$

Fact
if E_n is a predicate (that is $E_n \in \mathcal{H}$), then $E_n(n)$ is a sentence (by definition)

Definition
let A be a set over $\mathbb{N}$ (a number set), then

$$A^* := \{n \in \mathbb{N} \mid d(n) \in A\}$$

Theorem (Gödel's Theorem)
suppose $\mathcal{L}$ is correct and $P := \{g(S) \mid S \in \mathcal{P}\}$ such that $({\sim}P)^*$ is expressible in $\mathcal{L}$; then $\exists$ a true sentence of $\mathcal{L}$, not provable in $\mathcal{L}$

Proof.
on black board ∎

Fact
$({\sim}P)^*$ is expressible boils down to

- G1 *$\forall$ sets A expressible in $\mathcal{L}$, A^* is expressible in $\mathcal{L}$*
- G2 *$\forall$ sets A expressible in $\mathcal{L}$, ${\sim}A$ is expressible in $\mathcal{L}$*
- G3 *P is expressible in $\mathcal{L}$*

it will be simply to verify G1, it will be trivial to verify G2, but the hard part will be to verify G3

Gödel Sentences

Definition
E_n is a Gödel sentence for a number set A, if
1. either $E_n \in \mathcal{T}$ and $n \in A$, or
2. $E_n \notin \mathcal{T}$ and $n \notin A$

Lemma (Diagonal Lemma)
1. *$\forall$ sets A, if A^* is expressible in $\mathcal{L}$, then $\exists$ Gödel sentence for A*
2. *if $\mathcal{L}$ satisfies G1, then $\forall$ sets A expressible in $\mathcal{L}$, then $\exists$ Gödel sentence for A*

Proof.
on black board ∎

Theorem (Tarski's Theorem)

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ *is not nameable in* $\mathcal{L}$
2. *if G1 holds, then* $\sim T$ *ist not nameable in* $\mathcal{L}$
3. *if G1 & G2 hold, then* T *is not nameable in* $\mathcal{L}$

Proof.

assume ∃ Gödel sentence for $\sim T$; then ∃ sentence E_n such that E_n is true iff $n \notin T$; this is absurd

1. assume $(\sim T)^*$ is nameable, then ∃ Gödel sentence for $(\sim T)^*$; Contradiction
2. assume $\sim T$ is nameable, by G1 $(\sim T)^*$ is nameable; Contradiction
3. assume T is nameable, by G2 $\sim T$ is nameable; Contradiction

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Undecidable Sentences

Definition

1. $\mathcal{L}$ is consistent if ¬ ∃ sentence that is provable and refutable
2. a sentence S is decidable in $\mathcal{L}$ if it either provable or refutable; S is called undecidable otherwise
3. $\mathcal{L}$ is complete if ∀ sentences S, S is decidable; otherwise $\mathcal{L}$ is incomplete

Theorem

if $\mathcal{L}$ *is correct and if* $(\sim P)^*$ *is expressible in* $\mathcal{L}$*, then* $\mathcal{L}$ *is incomplete*

Proof.

by Gödel's Theorem ■

A Dual of Incompleteness

Theorem

suppose $\mathcal{L}$ *is correct and let* $R := \{g(S) \mid S \in \mathcal{S} \cap \mathcal{R}\}$

1. *suppose* R^* *is expressible in* $\mathcal{L}$*; then* $\mathcal{L}$ *is incomplete*
2. *moreover if* K *expresses* R^**, then its diagonalisation* $K(k)$ *is undecidable*

Proof.

1. if K expresses R^*, then

$$K(k) \text{ is true} \iff d(k) \in R$$

2. $d(k)$ is the Gödel number of $K(k)$
3. hence $K(k)$ is true iff it is refutable
4. thus $K(k)$ is undecidable

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