Gödel's Incompleteness Theorem

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Homework

- Suppose $\mathcal{L}$ is a correct system such that the following two conditions hold.
  1. The set P^* is expressible in $\mathcal{L}$.
  2. For any predicate H, there is a predicate H' such that for every n, the sentence $H'(n)$ is provable in $\mathcal{L}$ iff $H(n)$ is refutable in $\mathcal{L}$.

  Show that $\mathcal{L}$ is incomplete.

- We say that a predicate H *represents* a set A in $\mathcal{L}$ if for every number n, the sentence $H(n)$ is provable in $\mathcal{L}$ iff $n \in A$. Suppose $\mathcal{L}$ is consistent. Show that if the set R^* is representable in $\mathcal{L}$, then $\mathcal{L}$ is incomplete.

- Let us say that a predicate H *contrarepresents* of a set A in $\mathcal{L}$ if for every number n, the sentence $H(n)$ is refutable in $\mathcal{L}$ iff $n \in A$. Show that if the P^* is contrarepresentable in $\mathcal{L}$ and $\mathcal{L}$ is consistent, then $\mathcal{L}$ is incomplete.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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First Step

we study number theory based on addition, multiplication, and exponentiation

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Definition

the language $\mathcal{L}_E$ contains the following 13 symbols:

$$0 \quad ' \quad (\quad) \quad \mathsf{f} \quad , \quad \mathsf{v} \quad \neg \quad \to \quad \forall \quad = \quad \leqslant \quad \#$$

1. $0''''$ is a numeral and represents 4
2. $'$ represents the successor function
3. $\mathsf{f}_{,}$, $\mathsf{f}_{,,}$, $\mathsf{f}_{,,,}$ represents $+$, $\cdot$, exp
4. $\neg$, $\to$, $\forall$, $=$ are interpreted as usual
5. $\leqslant$ means "less than or equal"
6. $(\mathsf{v}_{,})$, $(\mathsf{v}_{,,})$, . . . represents variables v_1, v_2, . . .

Definition

terms are defined inductively :

1. variables ($\mathsf{v}_{\prime\ldots\prime}$) and numerals $0'\ldots'$ are terms
2. if s, t are terms, so are

$$\underbrace{\mathsf{f}_{\prime}(s \,,\, t)}_{(s+t)} \qquad \underbrace{\mathsf{f}_{\prime\prime}(s \,,\, t)}_{(s\cdot t)} \qquad \underbrace{\mathsf{f}_{\prime\prime\prime}(s \,,\, t)}_{(s \exp t)} \qquad s'$$

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Definition

$s = t$ or $s \leqslant t$ are atoms; formulas are defined inductively:

1. atoms are formulas
2. if A, B are formulas and v_i a variable, then

$$\neg A \qquad A \rightarrow B \qquad \forall v_i A$$

   are formulas

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1. free and bound variables are defined as usual
2. sentences or closed formulas of $\mathcal{L}_E$ are defined as usual
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5. $F(v_{i_1}, \dots, v_{i_n})$ is regular if $i_1 = 1, \dots i_n = n$
6. a regular formula can be written as $F(v_1, \dots, v_n)$

Definition

the degree of a formula is defined as follows:

$$\deg(F) := \begin{cases} 0 & F \text{ is an atom} \\ \deg(A) + 1 & (F = \neg A) \lor (F = \forall v_i A) \\ \deg(A) + \deg(B) + 1 & F = (A \to B) \end{cases}$$

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we use the following symbols as abbreviations

$$\begin{aligned} (A \vee B) &:= \dots & (A \wedge B) &:= \dots \\ (A \leftrightarrow B) &:= \dots & \exists v_i A &:= \dots \\ s \neq t &:= \dots & s < t &:= \dots \\ s^t &:= & (\forall v_i \leqslant t)F &:= \\ (\exists v_i \leqslant t)F &:= \end{aligned}$$

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$$s^t := s \exp t \qquad (\forall v_i \leqslant t)F := \forall v_i (v_i \leqslant t \to F)$$

$$(\exists v_i \leqslant t)F := \exists v_i (v_i \leqslant t \wedge F)$$

The Notion of Truth in $\mathcal{L}_E$

Definition

let $\mathcal{N}$ denote the standard model of number theory; the value of a closed term is defined as follows:

$$t^{\mathcal{N}} := \begin{cases} n & t = \overline{n} \\ {c_1}^{\mathcal{N}} + {c_2}^{\mathcal{N}} & t = (c_1 + c_2) \\ {c_1}^{\mathcal{N}} \cdot {c_2}^{\mathcal{N}} & t = (c_1 \cdot c_2) \\ ({c_1}^{\mathcal{N}})^{{c_2}^{\mathcal{N}}} & t = (c_1 \text{ exp } c_2) \\ c^{\mathcal{N}} + 1 & t = c' \end{cases}$$

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Example

consider the closed term

$$c := ((0''' + 0') \cdot (0'' \text{ exp } 0'''))'$$

Then $c^{\mathcal{N}} = (4 \cdot 2^3) + 1 = 33$

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Definition

an open formula $F(v_{i_1}, \ldots, v_{i_n})$ is said to be correct if the sentence $F(\overline{m}_1, \ldots, \overline{m}_n)$ is true for all numbers $m_1, \ldots, m_n$

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let $F(v_1)$ be $\exists v_2(v_2 \neq v_1)$, then what is $F(v_2)$?

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Recall

Gödel's argument is applicable to $\mathcal{L}$ if at least the following holds:

1. $\exists$ countable set of expressions $\mathcal{E}$
2. $\exists\ \mathcal{S} \subseteq \mathcal{E}$, $\mathcal{S}$ are the sentences
3. $\exists\ \mathcal{P} \subseteq \mathcal{S}$, the provable sentences
4. $\exists\ \mathcal{R} \subseteq \mathcal{S}$, the refutable sentences
5. $\exists\ \mathcal{H} \subseteq \mathcal{E}$, $\mathcal{H}$ are the predicates of $\mathcal{L}$, that is $H \in \mathcal{H}$ names a set of natural numbers
6. $\exists$ function Φ that maps expression E and number n to $E(n)$; for predicates $H(n)$ has to be a sentence: the sentences $H(n)$ expresses that n belongs to the set named by H
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2. $F(v_1, \dots, v_n)$ expresses R if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is true } \iff (m_1, \dots, m_n) \in R$$

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A be a set, and $R \subseteq \mathbb{N}^n$

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2. $F(v_1, \ldots, v_n)$ expresses R if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \ldots, \overline{m}_n) \text{ is true } \iff (m_1, \ldots, m_n) \in R$$

we also say that $F(v_1, \ldots, v_n)$ expresses the relation $R(x_1, \ldots, x_n)$

Tarski's Theorem

Definition

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Question ②

does it defeat Hilbert's program?

Concatenation and Gödel Numbering

Definition

let $b \geqslant 2$, we define the concatenation to the base b as follows:

$$m \ *_b \ n := m \cdot b^{|n|_b} + n$$

here m, n are numbers and $|n|_b$ denotes the length of the b-ary representation of n

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Lemma

for each $b \geqslant 2$, the relation $x *_b y = z$ is Arithmetic

Proof.

on white board ∎

Fact

$*_b$ is not associative:

$$(5 *_{10} 0) *_{10} 3 = 50 *_{10} 3 = 503 \qquad 5 *_{10} (0 *_{10} 3) = 5 *_{10} 3 = 53$$

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Corollary

for each $n \geqslant 2$ *and for each* $b \geqslant 2$*, the relation*

$$x_1 *_b x_2 *_b \cdots *_b x_n = z$$

is Arithmetic

Proof.

by induction on n from the previous lemma ∎