Gödel's Incompleteness Theorem

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Winter 2011

Homework

- Suppose $\mathcal{L}$ is a correct system such that the following two conditions hold.
  1. The set P^* is expressible in $\mathcal{L}$.
  2. For any predicate H, there is a predicate H' such that for every n, the sentence $H'(n)$ is provable in $\mathcal{L}$ iff $H(n)$ is refutable in $\mathcal{L}$.

  Show that $\mathcal{L}$ is incomplete.

- We say that a predicate H *represents* a set A in $\mathcal{L}$ if for every number n, the sentence $H(n)$ is provable in $\mathcal{L}$ iff $n \in A$.
  Suppose $\mathcal{L}$ is consistent. Show that if the set R^* is representable in $\mathcal{L}$, then $\mathcal{L}$ is incomplete.

- Let us say that a predicate H *contrarepresents* of a set A in $\mathcal{L}$ if for every number n, the sentence $H(n)$ is refutable in $\mathcal{L}$ iff $n \in A$. Show that if the P^* is contrarepresentable in $\mathcal{L}$ and $\mathcal{L}$ is consistent, then $\mathcal{L}$ is incomplete.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

The Language $\mathcal{L}_E$

First Step

we study number theory based on addition, multiplication, and exponentiation

Definition

the language $\mathcal{L}_E$ contains the following 13 symbols:

$$0 \quad ' \quad (\quad) \quad \mathrm{f} \quad , \quad \mathrm{v} \quad \neg \quad \rightarrow \quad \forall \quad = \quad \leqslant \quad \#$$

1. $0''''$ is a numeral and represents 4
2. $'$ represents the successor function
3. $\mathrm{f}_{,}$, $\mathrm{f}_{,,}$, $\mathrm{f}_{,,,}$ represents $+$, $\cdot$, exp
4. $\neg$, $\rightarrow$, $\forall$, $=$ are interpreted as usual
5. $\leqslant$ means "less than or equal"
6. $(\mathrm{v}_{,})$, $(\mathrm{v}_{,,})$, ... represents variables v_1, v_2, ...

Definition

terms are defined inductively :

1. variables $(\mathrm{v}_{\prime \ldots \prime})$ and numerals $0' \ldots '$ are terms
2. if s, t are terms, so are

$$\underbrace{\mathrm{f}_{\prime}(s \,{}_{\prime}\, t)}_{(s + t)} \qquad \underbrace{\mathrm{f}_{\prime\prime}(s \,{}_{\prime}\, t)}_{(s \cdot t)} \qquad \underbrace{\mathrm{f}_{\prime\prime\prime}(s \,{}_{\prime}\, t)}_{(s \ \exp \ t)} \qquad s'$$

terms without variables are called closed

Definition

$s = t$ or $s \leqslant t$ are atoms; formulas are defined inductively:

1. atoms are formulas
2. if A, B are formulas and v_i a variable, then

$$\neg A \qquad A \rightarrow B \qquad \forall v_i A$$

   are formulas

Definition

1. free and bound variables are defined as usual
2. sentences or closed formulas of $\mathcal{L}_E$ are defined as usual
3. an open formulas is a not-closed formula

we write $\overline{n}$ for the numeral $0' \ldots '$ designating n

Definition

let v_i be a variable and F a formula

1. $F(v_i)$ denotes a formula, where v_i is the only free variable
2. $F(\overline{n})$ denotes $F(v_i)\{v_i \mapsto \overline{n}\}$
3. for n free variables we write $F(v_{i_1}, \ldots, v_{i_n})$ and $F(\overline{m}_1, \ldots, \overline{m}_n)$
4. $F(\overline{m}_1, \ldots, \overline{m}_n)$ is instance of $F(v_{i_1}, \ldots, v_{i_n})$
5. $F(v_{i_1}, \ldots, v_{i_n})$ is regular if $i_1 = 1, \ldots i_n = n$
6. a regular formula can be written as $F(v_1, \ldots, v_n)$

Definition

the degree of a formula is defined as follows:

$$\deg(F) := \begin{cases} 0 & F \text{ is an atom} \\ \deg(A) + 1 & (F = \neg A) \vee (F = \forall v_i A) \\ \deg(A) + \deg(B) + 1 & F = (A \rightarrow B) \end{cases}$$

Definition

we use the following symbols as abbreviations

$$(A \vee B) := \ldots \qquad (A \wedge B) := \ldots$$
$$(A \leftrightarrow B) := \ldots \qquad \exists v_i A := \ldots$$
$$s \neq t := \ldots \qquad s < t := \ldots$$
$$s^t := s \exp t \qquad (\forall v_i \leqslant t)F := \forall v_i (v_i \leqslant t \rightarrow F)$$
$$(\exists v_i \leqslant t)F := \exists v_i (v_i \leqslant t \wedge F)$$

The Notion of Truth in $\mathcal{L}_E$

Definition

let $\mathcal{N}$ denote the standard model of number theory; the value of a closed term is defined as follows:

$$t^{\mathcal{N}} := \begin{cases} n & t = \overline{n} \\ {c_1}^{\mathcal{N}} + {c_2}^{\mathcal{N}} & t = (c_1 + c_2) \\ {c_1}^{\mathcal{N}} \cdot {c_2}^{\mathcal{N}} & t = (c_1 \cdot c_2) \\ ({c_1}^{\mathcal{N}})^{{c_2}^{\mathcal{N}}} & t = (c_1 \exp c_2) \\ c^{\mathcal{N}} + 1 & t = c' \end{cases}$$

Example

consider the closed term

$$c := ((0''' + 0') \cdot (0'' \exp 0'''))'$$

Then $c^{\mathcal{N}} = (4 \cdot 2^3) + 1 = 33$

Satisfaction Relation (Adapted)

Definition

let F be a sentence, $\mathcal{N} \models F$ is defined as:

$$\begin{aligned}
\mathcal{N} \models c_1 = c_2 &\iff \text{if } c_1{}^{\mathcal{N}} = c_2{}^{\mathcal{N}} \\
\mathcal{N} \models c_1 \leqslant c_2 &\iff \text{if } c_1{}^{\mathcal{N}} \leqslant c_2{}^{\mathcal{N}} \\
\mathcal{N} \models \neg A &\iff \text{if } \mathcal{N} \not\models A \\
\mathcal{N} \models A \to B &\iff \text{if } \mathcal{N} \models A\text{, then } \mathcal{N} \models B \\
\mathcal{N} \models \forall v_i A &\iff \text{if } \mathcal{N} \models A(\overline{n}) \text{ holds for all } n \in \mathbb{N}
\end{aligned}$$

if $\mathcal{N} \models F$, then F is true

Definition

an open formula $F(v_{i_1}, \dots, v_{i_n})$ is said to be correct if the sentence $F(\overline{m}_1, \dots, \overline{m}_n)$ is true for all numbers $m_1, \dots, m_n$

Substitution of Variables

Definition

consider a formula $F(v_1)$ and let $v_i \neq v_1$ be a variable; we define $F(v_i)$ as follows:

1. assume v_i is free for $F(v_1)$, then $F(v_i) := F(v_1)\{v_1 \mapsto v_i\}$
2. assume v_i is not free for $F(v_1)$
   - let v_j be variable that is free for $F(v_1)$ (such that j is minimal)
   - define $F'(v_1) := F\{v_i \mapsto v_j\}$
   - set $F(v_i) := F'(v_i)$, that is, we define $F(v_i) := F'(v_1)\{v_1 \mapsto v_i\}$

Example

let $F(v_1)$ be $\exists v_2(v_2 \neq v_1)$, then what is $F(v_2)$?

$\exists v_2(v_2 \neq v_2)$??? $\exists v_3(v_3 \neq v_2)$ ✓

Recall

Gödel's argument is applicable to $\mathcal{L}$ if at least the following holds:

1. $\exists$ countable set of expressions $\mathcal{E}$
2. $\exists\, \mathcal{S} \subseteq \mathcal{E}$, $\mathcal{S}$ are the sentences
3. $\exists\, \mathcal{P} \subseteq \mathcal{S}$, the provable sentences
4. $\exists\, \mathcal{R} \subseteq \mathcal{S}$, the refutable sentences
5. $\exists\, \mathcal{H} \subseteq \mathcal{E}$, $\mathcal{H}$ are the predicates of $\mathcal{L}$, that is $H \in \mathcal{H}$ names a set of natural numbers
6. $\exists$ function Φ that maps expression E and number n to $E(n)$; for predicates $H(n)$ has to be a sentence: the sentences $H(n)$ expresses that n belongs to the set named by H
7. $\exists\, \mathcal{T} \subseteq \mathcal{S}$, the true sentences

Question

what do we need to prove Tarski's theorem?

Definition

1. let A, B sentences, we say A and B are equivalent, if $A \models B$ and $B \models A$
2. let $A(v_{i_1}, \dots, v_{i_n})$, $B(v_{i_1}, \dots, v_{i_k})$ be formulas, we say they are equivalent, if all instances are equivalent

Definition

let $F(v_1)$ be a formula, $F(v_1, \dots, v_n)$ a regular formula, A be a set, and $R \subseteq \mathbb{N}^n$

1. $F(v_1)$ expresses A if for all $n \in \mathbb{N}$: $F(\overline{n})$ is true $\iff n \in A$
2. $F(v_1, \dots, v_n)$ expresses R if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is true} \iff (m_1, \dots, m_n) \in R$$

we also say that $F(v_1, \dots, v_n)$ expresses the relation $R(x_1, \dots, x_n)$

Tarski's Theorem

Definition
1. a set or relation is Arithmetic if expressible in $\mathcal{L}_E$
2. a set or relation is arithmetic if expressible in $\mathcal{L}_E$ without exp

Theorem
The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Question ①
does this imply that Gödel's theorem that there exists a true, but unprovable sentence?

Question ②
does it defeat Hilbert's program?

Concatenation and Gödel Numbering

Definition
let $b \geqslant 2$, we define the concatenation to the base b as follows:

$$m \ast_b n := m \cdot b^{|n|_b} + n$$

here m, n are numbers and $|n|_b$ denotes the length of the b-ary representation of n

Lemma
for each $b \geqslant 2$, the relation $x \ast_b y = z$ is Arithmetic

Proof.
on white board ■

Fact
$\ast_b$ is not associative:

$$(5 \ast_{10} 0) \ast_{10} 3 = 50 \ast_{10} 3 = 503 \qquad 5 \ast_{10} (0 \ast_{10} 3) = 5 \ast_{10} 3 = 53$$

so let's associate to the left

Corollary
for each $n \geqslant 2$ and for each $b \geqslant 2$, the relation

$$x_1 \ast_b x_2 \ast_b \cdots \ast_b x_n = z$$

is Arithmetic

Proof.
by induction on n from the previous lemma ■