Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

Winter 2011

Summary of Last Lecture

Definition

let $F(v_1)$ be a formula, $F(v_1, \dots, v_n)$ a regular formula,
A be a set, and $R \subseteq \mathbb{N}^n$

1. $F(v_1)$ expresses A if for all $n \in \mathbb{N}$: $F(\overline{n})$ is true $\iff$ $n \in A$
2. $F(v_1, \dots, v_n)$ expresses R if for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \dots, \overline{m}_n) \text{ is true} \iff (m_1, \dots, m_n) \in R$$

   we also say that $F(v_1, \dots, v_n)$ expresses the relation $R(x_1, \dots, x_n)$

Definition

1. a set or relation is Arithmetic if expressible in $\mathcal{L}_E$
2. a set or relation is arithmetic if expressible in $\mathcal{L}_E$ without exp

Homework

- For any set A of natural numbers and any function $f(x)$ (from natural numbers to natural numbers) by $f^{-1}(A)$, we mean the set of all n such that $f(n) \in A$. Prove that if A and f are Arithmetic, then so is $f^{-1}(A)$. Show the same for arithmetic.
-
  1. Given two Arithmetic functions $f(x)$ and $g(y)$, show that the function $f(g(y))$ is Arithmetic.
  2. Given two Arithmetic functions $f(x)$ and $g(x, y)$, show that the functions $g(f(y), y)$, $g(x, f(y))$ and $f(g(x, y))$ are all Arithmetic.
- Let A be an infinite Arithmetic set. Then for any number y (whether in A or not), there must be an element x of A which is greater than y. Let $R(x, y)$ be the relation: x is the smallest element of A greater than y. Prove that $R(x, y)$ is Arithmetic.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

$$0 \quad ' \quad (\quad) \quad \mathsf{f} \quad , \quad \mathsf{v} \quad \neg \quad \rightarrow \quad \forall \quad = \quad \leqslant \quad \#$$

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	$'$	(	)	f	,	v	$\neg$	$\rightarrow$	$\forall$	$=$	$\leqslant$	#
1	0	2	3	4	5	6	7	8	9	10	11	12

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	′	(	)	f	,	v	¬	→	∀	=	⩽	#
1	0	2	3	4	5	6	7	8	9	10	11	12
										η	ϵ	δ

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	$'$	(	)	f	,	v	$\neg$	$\rightarrow$	$\forall$	$=$	$\leqslant$	#
1	0	2	3	4	5	6	7	8	9	10	11	12
										η	ϵ	δ

2. for any expression E:

$$\ulcorner E \urcorner := \text{the concatenation of the Gödel numbers of the symbols to the base 13}$$

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	′	(	)	f	,	v	$\neg$	$\to$	$\forall$	$=$	$\leqslant$	#
1	0	2	3	4	5	6	7	8	9	10	11	12
										η	ϵ	δ

2. for any expression E:

$$\ulcorner E \urcorner := \text{the concatenation of the Gödel numbers of the symbols to the base 13}$$

3. E_n $(n > 0)$ denotes the expression with Gödel number n;

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	$'$	(	)	f	,	v	$\neg$	$\rightarrow$	$\forall$	$=$	$\leqslant$	#
1	0	2	3	4	5	6	7	8	9	10	11	12
										η	ϵ	δ

2. for any expression E:

$$\ulcorner E \urcorner := \text{the concatenation of the Gödel numbers of the symbols to the base 13}$$

3. E_n $(n > 0)$ denotes the expression with Gödel number n; $E_0 := {}'$

Gödel Numbering

Definition

1. to every symbol in $\mathcal{L}_E$ we assign a number $\leqslant 12$

0	′	(	)	f	,	v	¬	→	∀	=	⩽	#
1	0	2	3	4	5	6	7	8	9	10	11	12
										η	ϵ	δ

2. for any expression E:

$$\ulcorner E \urcorner := \text{the concatenation of the Gödel numbers of the symbols to the base 13}$$

3. E_n $(n > 0)$ denotes the expression with Gödel number n; $E_0 := {}'$

Example

consider the numeral $\overline{n}$:

$$\ulcorner \overline{n} \urcorner = \ulcorner 0' \cdots ' \urcorner = 1 *_{13} 0 *_{13} \cdots *_{13} 0 = 13^n$$

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1(v_1 = \overline{n} \rightarrow E)$

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1(v_1 = \overline{n} \rightarrow E)$
- as E is a formula, $E[\overline{n}]$ is a formula

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1(v_1 = \overline{n} \rightarrow E)$
- as E is a formula, $E[\overline{n}]$ is a formula
- if E is a formula, whose only free variable is v_1, then $E[\overline{n}]$ is even a sentence:

$$E[\overline{n}] = \forall v_1(v_1 = \overline{n} \rightarrow E(v_1))$$

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1(v_1 = \overline{n} \to E)$
- as E is a formula, $E[\overline{n}]$ is a formula
- if E is a formula, whose only free variable is v_1, then $E[\overline{n}]$ is even a sentence:

$$E[\overline{n}] = \forall v_1(v_1 = \overline{n} \to E(v_1))$$

- clearly $E(\overline{n})$ and $E[\overline{n}]$ are equivalent

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1(v_1 = \overline{n} \to E)$
- as E is a formula, $E[\overline{n}]$ is a formula
- if E is a formula, whose only free variable is v_1, then $E[\overline{n}]$ is even a sentence:

$$E[\overline{n}] = \forall v_1(v_1 = \overline{n} \to E(v_1))$$

- clearly $E(\overline{n})$ and $E[\overline{n}]$ are equivalent

Definition (representation function)

- set $r(e, n) := \ulcorner E[\overline{n}] \urcorner$, where $\ulcorner E \urcorner = e$

A Clever Trick by Tarski

Definition (Tarski's Trick)

let E be an formula and $e, n \in \mathbb{N}$

- set $E[\overline{n}] := \forall v_1 (v_1 = \overline{n} \rightarrow E)$
- as E is a formula, $E[\overline{n}]$ is a formula
- if E is a formula, whose only free variable is v_1, then $E[\overline{n}]$ is even a sentence:

$$E[\overline{n}] = \forall v_1 (v_1 = \overline{n} \rightarrow E(v_1))$$

- clearly $E(\overline{n})$ and $E[\overline{n}]$ are equivalent

Definition (representation function)

- set $r(e, n) := \ulcorner E[\overline{n}] \urcorner$, where $\ulcorner E \urcorner = e$
- thus the representation function $r(x, y)$ is the Gödel number of $E_x[\overline{y}]$

Lemma

the function $r(x, y)$ is Arithmetic

Proof.

on the white board ∎

Lemma

the function $r(x, y)$ *is Arithmetic*

Proof.

on the white board ∎

Definition

- we define a concrete diagonal function: $d(x) := r(x, x)$

Lemma

the function $r(x, y)$ is Arithmetic

Proof.

on the white board ■

Definition

- we define a concrete diagonal function: $d(x) := r(x, x)$
- for any set A, we define $A^* := \{n \in \mathbb{N} \mid d(n) \in A\}$ (as in the abstract setting)

Lemma

the function $r(x, y)$ is Arithmetic

Proof.

on the white board ∎

Definition

- we define a concrete diagonal function: $d(x) := r(x, x)$
- for any set A, we define $A^* := \{n \in \mathbb{N} \mid d(n) \in A\}$ (as in the abstract setting)

Lemma ①

if A is Arithmetic, then so is A^*

Proof.

on the white board ∎

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

1. suppose A is Arithmetic

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

1. suppose A is Arithmetic
2. by Lemma ①, A^* is Arithmetic

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

1. suppose A is Arithmetic
2. by Lemma ①, A^* is Arithmetic
3. suppose $H(v_1)$ expresses A^* and let $h := \ulcorner H \urcorner$

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

1. suppose A is Arithmetic
2. by Lemma ①, A^* is Arithmetic
3. suppose $H(v_1)$ expresses A^* and let $h := \ulcorner H \urcorner$
4. hence we obtain:

$$H[\overline{h}] \text{ is true} \iff h \in A^* \iff d(h) \in A \iff \ulcorner H[\overline{h}] \urcorner \in A$$

Recall

E_n is a Gödel sentence for a number set A, if

$$E_n \text{ holds} \iff n \in A$$

Theorem ①

for every Arithmetic set A, there is a Gödel sentence for A

Proof.

1. suppose A is Arithmetic
2. by Lemma ①, A^* is Arithmetic
3. suppose $H(v_1)$ expresses A^* and let $h := \ulcorner H \urcorner$
4. hence we obtain:

$$H[\overline{h}] \text{ is true} \iff h \in A^* \iff d(h) \in A \iff \ulcorner H[\overline{h}] \urcorner \in A$$

5. we conclude that $H[\overline{h}]$ is a Gödel sentence for A

■

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T
2. then $\neg F(v_1)$ expresses $\sim T$, and $\sim T$ is Arithmetic

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T
2. then $\neg F(v_1)$ expresses $\sim T$, and $\sim T$ is Arithmetic
3. hence there exists a Gödel sentence for $\sim T$

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T
2. then $\neg F(v_1)$ expresses $\sim T$, and $\sim T$ is Arithmetic
3. hence there exists a Gödel sentence for $\sim T$
4. let E_n be a Gödel sentence of $\sim T$, that is E_n holds iff $n \notin T$

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T
2. then $\neg F(v_1)$ expresses $\sim T$, and $\sim T$ is Arithmetic
3. hence there exists a Gödel sentence for $\sim T$
4. let E_n be a Gödel sentence of $\sim T$, that is E_n holds iff $n \notin T$
5. this is absurd and we arrive at a contradiction

Tarski's Theorem

Theorem

The set T of Gödel numbers of the true Arithmetic sentences is not Arithmetic

Proof.

we argue indirectly

1. suppose T is Arithmetic, that is, there exists a formula $F(v_1)$ that expresses T
2. then $\neg F(v_1)$ expresses $\sim T$, and $\sim T$ is Arithmetic
3. hence there exists a Gödel sentence for $\sim T$
4. let E_n be a Gödel sentence of $\sim T$, that is E_n holds iff $n \notin T$
5. this is absurd and we arrive at a contradiction

∎

The Abstract Framework (revisited)

Discussion ①

compare to the abstract framework:

- $\mathcal{E}$ are expressions of $\mathcal{L}_E$
- $\mathcal{S}$ are sentences of $\mathcal{L}_E$
- $\mathcal{H}$ are formulas $F(v_1)$, where only v_1 is free
- $\Phi(E, n) := E[\overline{n}]$
- $\mathcal{T}$ are the true sentences of $\mathcal{L}_E$
- $g(\cdot)$ becomes $\ulcorner \cdot \urcorner$

The Abstract Framework (revisited)

Discussion ①

compare to the abstract framework:

- $\mathcal{E}$ are expressions of $\mathcal{L}_E$
- $\mathcal{S}$ are sentences of $\mathcal{L}_E$
- $\mathcal{H}$ are formulas $F(v_1)$, where only v_1 is free
- $\Phi(E, n) := E[\overline{n}]$
- $\mathcal{T}$ are the true sentences of $\mathcal{L}_E$
- $g(\cdot)$ becomes $\ulcorner\cdot\urcorner$

Recall

G1 $\forall$ sets A expressible in $\mathcal{L}$, A^* is expressible in $\mathcal{L}$

G2 $\forall$ sets A expressible in $\mathcal{L}$, $\sim A$ is expressible in $\mathcal{L}$

Tarski's Theorem in the Abstract Framework

Recall

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ is not nameable in $\mathcal{L}$
2. if G1 holds, then $\sim T$ is not nameable in $\mathcal{L}$
3. if G1 & G2 hold, then T is not nameable in $\mathcal{L}$

Tarski's Theorem in the Abstract Framework

Recall

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ is not nameable in $\mathcal{L}$
2. if G1 holds, then $\sim T$ is not nameable in $\mathcal{L}$
3. if G1 & G2 hold, then T is not nameable in $\mathcal{L}$

Discussion ②

observe that

Tarski's Theorem in the Abstract Framework

Recall

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ is not nameable in $\mathcal{L}$
2. if G1 holds, then $\sim T$ is not nameable in $\mathcal{L}$
3. if G1 & G2 hold, then T is not nameable in $\mathcal{L}$

Discussion ②

observe that

1. property $G1$ is expressed by Lemma ① and property $G2$ is trivial for the set of Arithmetic sentences

Tarski's Theorem in the Abstract Framework

Recall

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ is not nameable in $\mathcal{L}$
2. if G1 holds, then $\sim T$ is not nameable in $\mathcal{L}$
3. if G1 & G2 hold, then T is not nameable in $\mathcal{L}$

Discussion ②

observe that

1. property $G1$ is expressed by Lemma ① and property $G2$ is trivial for the set of Arithmetic sentences
2. Theorem ① is the second part of the Diagonal Lemma

Tarski's Theorem in the Abstract Framework

Recall

let $T := \{g(S) \mid S \in \mathcal{T}\}$

1. $(\sim T)^*$ is not nameable in $\mathcal{L}$
2. if G1 holds, then $\sim T$ is not nameable in $\mathcal{L}$
3. if G1 & G2 hold, then T is not nameable in $\mathcal{L}$

Discussion ②

observe that

1. property $G1$ is expressed by Lemma ① and property $G2$ is trivial for the set of Arithmetic sentences
2. Theorem ① is the second part of the Diagonal Lemma

thus Tarski's Theorem for $\mathcal{L}_E$ is nothing but an instance of the abstract form of Tarski's Theorem

The Axiom System PE

Definition (Propositional Logic)

$$L_1: \quad F \rightarrow (G \rightarrow F)$$

$$L_2: \quad F \rightarrow (G \rightarrow H)) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow H)$$

$$L_3: \quad (\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$$

The Axiom System PE

Definition (Propositional Logic)

$$L_1: \quad F \rightarrow (G \rightarrow F)$$

$$L_2: \quad F \rightarrow (G \rightarrow H)) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow H)$$

$$L_3: \quad (\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$$

Definition (First-Order Logic with Identity)

$$L_4: \quad \forall v_i (F \rightarrow G) \rightarrow (\forall v_i F \rightarrow \forall v_i G)$$

$$L_5: \quad F \rightarrow \forall v_i F$$

$$L_6: \quad \exists v_i (v_i = t)$$

$$L_7: \quad v_i = t \rightarrow (X_1 v_i X_2 \rightarrow X_1 t X_2)$$

where v_i doesn't occur in F or in t and X_1, X_2 are expressions, such that $X_1 v_i X_2$ is an atom

The Axiom System PE (cont'd)

Definition (Axioms of Arithmetic)

$$
\begin{aligned}
N_1 &: \quad v_1' = v_2' \rightarrow v_1 = v_2 \\
N_2 &: \quad \overline{0} \neq v_1' \\
N_3 &: \quad (v_1 + \overline{0}) = v_1 \\
N_4 &: \quad (v_1 + v_2') = (v_1 + v_2)' \\
N_5 &: \quad (v_1 \cdot \overline{0}) = \overline{0} \\
N_6 &: \quad (v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1) \\
N_7 &: \quad (v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0}) \\
N_8 &: \quad (v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \vee v_1 = v_2') \\
N_9 &: \quad (v_1 \leqslant v_2) \vee (v_2 \leqslant v_1) \\
N_{10} &: \quad (v_1 \ \exp\ \overline{0}) = \overline{0}' \\
N_{11} &: \quad (v_1 \ \exp\ v_2') = ((v_1 \ \exp\ v_2) \cdot v_1)
\end{aligned}
$$

The Axiom System PE (cont'd)

Definition (Induction Schema)

The Axiom System PE (cont'd)

Definition (Induction Schema)

1. let $F(v_1)$ denote a formula with a free variable v_1

The Axiom System PE (cont'd)

Definition (Induction Schema)

1. let $F(v_1)$ denote a formula with a free variable v_1
2. let $F[v_1']$ denote any of

$$\forall v_i(v_i = v_1' \rightarrow \forall v_1(v_1 = v_i \rightarrow F)$$

   where v_i doesn't occur in F

The Axiom System PE (cont'd)

Definition (Induction Schema)

1. let $F(v_1)$ denote a formula with a free variable v_1
2. let $F[v_1']$ denote any of

$$\forall v_i(v_i = v_1' \rightarrow \forall v_1(v_1 = v_i \rightarrow F)$$

where v_i doesn't occur in F

then

$$N_{12}\colon \qquad F[\overline{0}] \rightarrow (\forall v_1(F(v_1) \rightarrow F[v_1']) \rightarrow \forall v_1 F(v_1))$$

The Axiom System PE (cont'd)

Definition (Induction Schema)

1. let $F(v_1)$ denote a formula with a free variable v_1
2. let $F[v_1']$ denote any of

$$\forall v_i(v_i = v_1' \to \forall v_1(v_1 = v_i \to F)$$

where v_i doesn't occur in F

then

$$N_{12}: \qquad F[\overline{0}] \to (\forall v_1(F(v_1) \to F[v_1']) \to \forall v_1 F(v_1))$$

Definition (Inference Rules)

$$\frac{F \to G \quad F}{G} \text{ Modus Ponens} \qquad \frac{F}{\forall v_i F} \text{ Generalisation}$$