Gödel's Incompleteness Theorem

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Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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Arithmetisation

$$xB_by \leftrightarrow x = y \vee (x \neq 0 \wedge (\exists z \leqslant y)(\exists w \leqslant y)$$
$$(\mathrm{Pow}_b(w) \wedge (x \cdot w) *_b z = y))$$
$$xE_by \leftrightarrow x = y \vee (\exists z \leqslant y)(z *_b x = y)$$
$$xP_by \leftrightarrow (\exists z \leqslant y)(zE_by \wedge xB_bz)$$

Lemma

for any $b \geqslant 2$, $n \geqslant 2$

1. *the relations xB_by, xE_by, and xP_by are Arithmetic*
2. *furthermore the following relation is Arithmetic*

$$x_1 *_b x_2 *_b \cdots *_b x_n P_b y \leftrightarrow$$
$$(\exists z \leqslant y)(x_1 *_b x_2 *_b \cdots *_b x_n = z) \wedge (z P_b y)$$

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Notation

in the following we fix $b = 13$ and simply write

1. xBy, xEy, xPy
2. xy instead of $x *_{13} y$
3. $x_1 \cdots x_n Py$ for $x_1 *_{13} \cdots *_{13} x_n Py$

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- $x \ \in \ y$ denotes that x is a member of a sequence encoded by y
- $x \ \prec_z \ y$ denotes that $x \in z$, $y \in z$ and x occurs first

Formation Sequences

Lemma

the relations $\mathsf{Seq}\, x$, $x \in y$, $x \prec_z y$ *are Arithmetic*

Proof.

on the whiteboard ∎

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Some More Notation

we write $(\forall x \in y)$ [some formula] instead of

$$\forall x\, (x \in y \to \text{[some formula]})$$

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and we write $(\exists x, y \prec_w z)$ [some formula] instead of

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Definition (Formulas (explicit))

(as above)

Towards Expressing Provability

And Even More Notation

we refer to the Gödel numbers of $(E_x + E_y)$, $(E_x \cdot E_y)$, $(E_x \text{ exp } E_y)$, E'_x as

$$x \text{ pl } y \qquad x \text{ tim } y \qquad x \text{ expon } y \qquad \mathsf{s}(x)$$

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and to the Gödel numbers of $E_x = E_y$, $E_x \leqslant E_y$, $\neg E_x$, and $(E_x \to E_y)$ as

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Lemma

all above functions are Arithmetic

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Definition

1. $\text{Sb}(x)$: E_x is a string of subscripts

$$(\forall y \leqslant x)\,(yPx \rightarrow 5Py)$$

Definition (Terms (formal))

1. Var(x): E_x is a variable

$$(\exists y \leqslant x)\,(\mathrm{Sb}(y) \wedge x = 26y3)$$

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3. $\mathrm{R}_1(x, y, z)$: $\mathcal{R}_\mathrm{t}(E_x, E_y, E_z)$ holds

$$(z = x \text{ pl } y) \vee (z = x \text{ tim } y) \vee (z = x \text{ expon } y) \vee (z = \mathrm{s}(x))$$

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4. $\mathrm{Seqt}(x)$: E_x is a formation sequence for terms

$$\mathrm{Seq}(x) \land (\forall y \in x)(\mathrm{Var}(y) \lor \mathrm{Num}(y) \lor (\exists z, w \prec_x y)\,\mathrm{R}_1(z, w, y))$$

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5. term(x): E_x is a term

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Definition (Provable and Refutable)

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5. $\mathrm{P_E}(x)$: E_x is provable in PE: $\exists y(\mathrm{Proof}(y) \wedge x \in y)$
6. $\mathrm{R_E}(x)$: E_x is refutable in PE: $\mathrm{P_E}(\mathrm{neg}(x))$

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all conditions are Arithmetic

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Theorem

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Proof.

- let P_E (R_E) denote the set of Gödel numbers of provable (refutable) sentences of PE
- let $P(v_1)$ ($R(v_1)$) express these sets in $\mathcal{L}_E$
- $\neg P(v_1)$ expresses $\sim P_E$
- by Lemma ① $\exists$ $H(v_1)$ that expresses $(\sim P_E)^*$
- by Theorem ① $H[\overline{h}]$ is Gödel sentence of $(\sim P_E)^*$
- hence $H[\overline{h}]$ is neither provable nor refutable

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