Gödel's Incompleteness Theorem

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Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Arithmetisation of the Axiom System

Definition

- x begins y in base b notation if — xB_by
  base b notation of x is prefix of base b notation of y
- x ends y in base b notation if — xE_by
  base b notation of x is suffix of base b notation of y
- x is part of y in base b notation if — xP_by
  if x ends some number that begins y

Arithmetisation

$$xB_by \leftrightarrow x = y \lor (x \neq 0 \land (\exists z \leqslant y)(\exists w \leqslant y)$$
$$(\mathrm{Pow}_b(w) \land (x \cdot w) *_b z = y))$$
$$xE_by \leftrightarrow x = y \lor (\exists z \leqslant y)(z *_b x = y)$$
$$xP_by \leftrightarrow (\exists z \leqslant y)(zE_by \land xB_bz)$$

Lemma

for any $b \geqslant 2$, $n \geqslant 2$

1. *the relations xB_by, xE_by, and xP_by are Arithmetic*
2. *furthermore the following relation is Arithmetic*

$$x_1 *_b x_2 *_b \cdots *_b x_n P_b y \leftrightarrow$$
$$(\exists z \leqslant y)(x_1 *_b x_2 *_b \cdots *_b x_n = z) \land (zP_by)$$

Notation

in the following we fix $b = 13$ and simply write

1. xBy, xEy, xPy
2. xy instead of $x *_{13} y$
3. $x_1 \cdots x_n Py$ for $x_1 *_{13} \cdots *_{13} x_n Py$

Representing Sequences

Definition (Formal Finite Sequences)

- we represent the tuple $(X_1, \dots, X_n)$ as $\#X_1\# \cdots \#X_n\#$
- $\ulcorner \#X_1\# \cdots \#X_n\# \urcorner$ will be the sequence number of the tuple
- we define K_{11}:
$$K_{11} := \{n \in \mathbb{N} \mid (n)_{13} \text{ does not contain } \delta\}$$
- for $(a_1, \dots, a_n) \in K_{11}^n$, define
$$\delta a_1 \delta \cdots \delta a_n \delta$$
as the sequence number of $(a_1, \dots, a_n)$
- x is a sequence number if $x = \delta a_1 \delta \cdots \delta a_n \delta$ for $a_i \in K_{11}$
- $\operatorname{Seq} x$ denotes that x is a sequence number
- $x \in y$ denotes that x is a member of a sequence encoded by y
- $x \prec_z y$ denotes that $x \in z$, $y \in z$ and x occurs first

Formation Sequences

Lemma

the relations $\operatorname{Seq} x$, $x \in y$, $x \prec_z y$ *are Arithmetic*

Proof.

on the whiteboard ∎

Some More Notation

we write $(\forall x \in y)$ [some formula] instead of
$$\forall x\, (x \in y \to [\text{some formula}])$$
and we write $(\exists x, y \prec_w z)$ [some formula] instead of
$$\exists x \exists y\, (x \prec_w z \land y \prec_w z \land [\text{some formula}])$$

Definition (Terms (explicit))

- for expressions X, Y, Z we define $\mathcal{R}_t(X, Y, Z)$ iff
$$Z = (X + Y) \lor Z = (X \cdot Y) \lor Z = (X \exp Y) \lor Z = X'$$
- $\mathcal{R}_t(X, Y, Z)$ is called the formation relation for terms
- a formation sequence for terms is a finite sequence of expressions
$$X_1, X_2, \dots, X_n$$
such that X_i is
  1. either a variable $((v'^{\cdots'}))$ or a numeral $(0' \dots ')$
  2. or $\exists\ X_j, X_k\ (j, k < i)$ such that $\mathcal{R}_t(X_j, X_k, X_i)$ holds
- an expression t is a term, if $\exists$ a formation sequence for terms of which t is a member

Definition (Formulas (explicit))

(as above)

Towards Expressing Provability

And Even More Notation

we refer to the Gödel numbers of $(E_x + E_y)$, $(E_x \cdot E_y)$, $(E_x \exp E_y)$, E_x' as
$$x \text{ pl } y \qquad x \text{ tim } y \qquad x \text{ expon } y \qquad \mathrm{s}(x)$$
and to the Gödel numbers of $E_x = E_y$, $E_x \leqslant E_y$, $\neg E_x$, and $(E_x \to E_y)$ as
$$x \text{ id } y \qquad x \text{ le } y \qquad \mathrm{neg}(x) \qquad x \text{ impl } y$$

Lemma

all above functions are Arithmetic

Definition

1. $\mathrm{Sb}(x)$: E_x is a string of subscripts
$$(\forall y \leqslant x)\, (yPx \to 5Py)$$

Definition (Terms (formal))

1. Var(x): E_x is a variable
$$(\exists y \leqslant x)\,(\mathrm{Sb}(y) \wedge x = 26y3)$$
2. Num(x): E_x is a numeral
$$\mathrm{Pow}_{13}(x)$$
3. $\mathrm{R}_1(x,y,z)$: $\mathcal{R}_\mathrm{t}(E_x, E_y, E_z)$ holds
$$(z = x \text{ pl } y) \vee (z = x \text{ tim } y) \vee (z = x \text{ expon } y) \vee (z = \mathrm{s}(x))$$
4. Seqt(x): E_x is a formation sequence for terms
$$\mathrm{Seq}(x) \wedge (\forall y \in x)(\mathrm{Var}(y) \vee \mathrm{Num}(y) \vee (\exists z, w \prec_x y)\, \mathrm{R}_1(z,w,y))$$
5. term(x): E_x is a term
$$\exists y(\mathrm{Seqt}(y) \wedge x \in y)$$

Definition (Formulas (formal))

1. atom(x): E_x is an atom
$$(\exists y \leqslant x)(\exists z \leqslant x)\,(\mathrm{term}(y) \wedge \mathrm{term}(z) \wedge (x = y \text{ id } z \vee x = y \text{ le } z))$$
2. Gen(x, y): $E_y = \forall w E_x$ for some variable w
$$(\exists z \leqslant y)(\mathrm{Var}(z) \wedge y = 9zx)$$
3. R2(x,y,z): $\mathcal{R}_\mathrm{f}(E_x, E_y, E_z)$ holds
$$(z = \mathrm{neg}(x)) \vee (z = x \text{ impl } y) \vee \mathrm{Gen}(x,z)$$
4. Seqf(x): E_x is a formation sequence for formulas
$$\mathrm{Seq}(x) \wedge (\forall y \in x)\,(\mathrm{atom}(y) \vee (\exists z, w \prec_x y)\, \mathrm{R2}(z,w,y))$$
5. formula(x): E_x is a formula
$$\exists y(\mathrm{Seqf}(y) \wedge x \in y)$$

Definition (Provable and Refutable)

1. Axiom(x): E_x is an axiom

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2. MP(x,y,z): E_z follows by Modus Ponens from E_x and E_y
$$y = x \text{ impl } z$$
3. Deriv(x,y,z): E_z is derivable from E_x and E_y
$$\mathrm{MP}(x,y,z) \vee \mathrm{Gen}(x,z)$$
4. Proof(x): E_x is a proof in PE
$$\mathrm{Seq}(x) \wedge (\forall y \in x)\,(\mathrm{Axiom}(y) \vee (\exists z, w \prec_x y)\, \mathrm{Deriv}(z,w,y))$$
5. $\mathrm{P_E}(x)$: E_x is provable in PE: $\exists y(\mathrm{Proof}(y) \wedge x \in y)$
6. $\mathrm{R_E}(x)$: E_x is refutable in PE: $\mathrm{P_E}(\mathrm{neg}(x))$

Lemma

all conditions are Arithmetic

Theorem

the axiom system PE *is incomplete*

Proof.

- let P_E (R_E) denote the set of Gödel numbers of provable (refutable) formulas of PE
- let $P(v_1)$ ($R(v_1)$) express these sets in $\mathcal{L}_E$
- $\neg P(v_1)$ expresses $\sim P_E$
- by Lemma ① $\exists$ $H(v_1)$ that expresses $(\sim P_E)^*$
- by Theorem ① $H[\overline{h}]$ is Gödel sentence of $(\sim P_E)^*$
- hence $H[\overline{h}]$ is neither provable nor refutable

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